Global Conharmonic Concept Type Nearly Kahler Manifold

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Abstract

The concept of permanence conharmonic type Nearly Kahler manifold conditions are obtained when the Nearly Kahler manifold is a manifold conharmonic constant type. Proved that the local conharmonic constancy of type Nearly Kahler manifold is equivalent to its global conharmonic constancy of type. And also proved that the Nearly Kahler manifold conharmonic constant type is a manifold of constant scalar curvature. The notion of constancy of type Nearly Kahler manifolds was introduced A.Greem ([1], [2], [3]) and has proved very useful in the study of the geometry of nearly kahler manifolds. Next nearly kahler manifolds of constant type considered various mathematicians (see. [1], [3], [4], [5], [6], [7]). Comprehensive description of Nearly Kahler manifolds of constant type was obtained V.F.Kirichenko [6]. In [6] it is shown that Nearly Kahler manifold of pointwise constant type description by the identity $B_{abh}B^{hcd} = B\delta^{cd}_{ab}$, where B - a function on Nearly Kahler manifold, $\delta^{cd}_{ab} = \delta^c_a\delta^d_b - \delta^d_a\delta^c_b$. Wherein local constancy of type Nearly Kahler manifold is equivalent to the local constancy of its type. Moreover, the covariant constancy of structure tensors of the first and second kind it follows immediately that, the constancy of type at only one point Nearly Kahler space implies global consistency of its type.

Keywords: Conharmonic concept type nearly kahler manifold, manifold concept scalar curvature.

Recall the definitions

Let M^{2n} - Nearly Kahler manifold with AH - structure {g, J} ∇ - Riemannian connection of the metric g.

Definition 1. [3]. Nearly Kahler manifold M^{2n} has constant type at a point $p \in M^{2n}$ if $\forall X,Y,Z \in T(M^{2n})$: $<X,Y>=<JX,Y>=<X,Z>=0$;
$\|Y\| = \|Z\| \Rightarrow \|\nabla_X(J)(Y)\| = \|\nabla_X(J)(Z)\|$. (Here $T(M^{2n})$ - tangent space M^{2n} in $p \in M^{2n}$.)

Definition 2 [3]. Nearly Kahler -manifold having a constant type at each point, called NK-manifold of pointwise constant type.

Definition 3 [3]. Nerely Kahler manifold M^{2n} pointwise constant type is globally constant type if $\forall X,Y \in X(M^{2n})$: $<X,Y>=<JX,Y>=0$; $\|X\| = \|Y\| = 1 \Rightarrow \|\nabla_X(J)(Y)\| = const$.

These concepts have been summarized L. Vanhecke and Boutenom for arbitrary almost Hermitian manifolds [8].

As shown by L. Vanhecke, if the almost Hermitian manifold belongs to Nearly Kahler manifolds, these definitions are reduced to the above definitions A.Greya.

It is now natural to introduce the definition of constant type, similar definitions Vanhecke, replacing the tensor R Riemann-Christoffel tensor conharmonic curvature.

We recall some useful formulas, we need, is easily obtained from the first group of structural equations Nearly Kahler manifolds [6], [7]:

$$\begin{aligned}
&1)\ dA^a_b = dA^{ah}_{bh} = A^{ah}_{bhc}\omega^c + A^{ahc}_{bh}\omega_c + A^a_c\theta^c_b - A^c_b\theta^a_c;\\
&2)\ dB^{ad}_{bc} = -B^{hd}_{bc}\theta^a_h - B^{ah}_{bc}\theta^d_h + B^{ad}_{hc}\theta^h_b + B^{ad}_{bh}\theta^h_c;\\
&3)\ dB^a_b = B^a_c\theta^c_b - B^c_b\theta^a_c.
\end{aligned} \quad (2)$$

Theorem 1. Nearly Kahler manifold is a manifold conharmonic constant type if and only if

$$B^{abh}B_{hcd} + \frac{1}{4(n-1)}\{\delta^a_d\left(A^b_c + 3B^b_c\right) - \delta^a_c\left(A^b_d + 3B^b_d\right) + \delta^b_c\left(A^a_d + 3B^a_d\right) - \delta^b_d\left(A^a_c + 3B^a_c\right)\} = -\frac{1}{2}c\delta^{ab}_{cd}.$$

The proof:

Let M^{2n} - Nerely Kahler manifold. To calculate the (X, Y, Z, W) and K (X, Y, JX, JY) on the space of the associated G-structure:

$$\begin{aligned}
K(X,Y,Z,W) = K_{ijkl}X^iY^jZ^kW^l = K_{\hat{a}bc\hat{d}}X^{\hat{a}}Y^bZ^cW^{\hat{d}} &+ K_{\hat{a}b\hat{c}d}X^{\hat{a}}Y^bZ^{\hat{c}}W^d + K_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}Z^cW^d + K_{a\hat{b}c\hat{d}}X^aY^{\hat{b}}Z^cW^{\hat{d}} +\\
&+ K_{a\hat{b}c\hat{d}}X^aY^{\hat{b}}Z^cW^{\hat{d}} + K_{ab\hat{c}\hat{d}}X^aY^bZ^{\hat{c}}W^{\hat{d}};
\end{aligned}$$

$$\begin{aligned}
K(X,Y,JX,JY) = K_{ijkl}X^iY^j(JX)^k(JY)^l = K_{\hat{a}bc\hat{d}}X^{\hat{a}}Y^bX^cY^{\hat{d}} &+ K_{\hat{a}b\hat{c}d}X^{\hat{a}}Y^bX^{\hat{c}}Y^d - K_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}X^cY^d + K_{a\hat{b}c\hat{d}}X^aY^{\hat{b}}X^cY^{\hat{d}} +\\
&+ K_{a\hat{b}\hat{c}d}X^aY^{\hat{b}}X^{\hat{c}}Y^d - K_{ab\hat{c}\hat{d}}X^aY^bX^{\hat{c}}Y^{\hat{d}}.
\end{aligned}$$

We show,

$$K(X,Y,JX,JY) - K(X,Y,X,Y) = -4K_{\hat{a}\hat{b}cd}X^{\hat{a}}Y^{\hat{b}}X^cY^d$$

In this way,. Given the specter tensor of curvature conharmonic last equation can be written as [9]:

$$K(X,Y,JX,JY) - K(X,Y,X,Y) = [-8B^{ab}_{cd} - \frac{2}{n-1}\{\delta^a_d\left(A^b_c + 3B^b_c\right) - \delta^a_c\left(A^b_d + 3B^b_d\right) + \delta^b_c\left(A^a_d + 3B^a_d\right) - \delta^b_d\left(A^a_c + 3B^a_c\right)\}]X^{\hat{a}}Y^{\hat{b}}X^cY^d.$$

Since $c\|X\|^2\|Y\|^2 = 4c\delta^{ab}_{cd}X^cY^dX_aY_b$ then

$$-8B^{ab}_{cd} - \frac{2}{n-1}\{\delta^a_d\left(A^b_c + 3B^b_c\right) - \delta^a_c\left(A^b_d + 3B^b_d\right) + \delta^b_c\left(A^a_d + 3B^a_d\right) - \delta^b_d\left(A^a_c + 3B^a_c\right)\} = 4c\delta^{ab}_{cd},$$

ie

$$B^{abh}B_{hcd}+\frac{1}{4(n-1)}\{\delta_d^a\left(A_c^b+3B_c^b\right)-\delta_c^a\left(A_d^b+3B_d^b\right)+$$
$$+\delta_c^b\left(A_d^a+3B_d^a\right)-\delta_d^b\left(A_c^a+3B_c^a\right)\}=-\frac{1}{2}c\delta_{cd}^{ab}. \quad (1)$$

This theorem proved

Theorem 2. Local conharmonic constant type Nearly Kahler manifold is equivalent to its global conharmonic constant type.

The proof:

We differentiate externally with regard to (2) Equation (1):

$$-B^{fdh}B_{abh}\theta_f^c-B^{cfh}B_{abh}\theta_f^d+B^{cdh}B_{fbh}\theta_a^f+B^{cdh}B_{bfh}\theta_b^f+ \quad (3)$$
$$+\frac{1}{4(n-1)}\delta_a^c\left\{A_{bhg}^{dh}\omega^g+A_{bh}^{dhg}\omega_g+\left(A_f^d+3B_f^d\right)\theta_b^f-\left(A_b^f+3B_b^f\right)\theta_f^d\right\}+$$
$$+\frac{1}{4(n-1)}\delta_b^d\left\{A_{ahg}^{ch}\omega^g+A_{ah}^{chg}\omega_g+\left(A_f^c+3B_f^c\right)\theta_a^f-\left(A_a^f+3B_a^f\right)\theta_f^c\right\}-$$
$$-\frac{1}{4(n-1)}\delta_a^d\left\{A_{bhg}^{ch}\omega^g+A_{bh}^{chg}\omega_g+\left(A_f^c+3B_f^c\right)\theta_b^f-\left(A_b^f+3B_b^f\right)\theta_f^c\right\}-$$
$$-\frac{1}{4(n-1)}\delta_b^c\left\{A_{ahg}^{dh}\omega^g+A_{ah}^{dhg}\omega_g+\left(A_f^d+3B_f^d\right)\theta_a^f-\left(A_a^f+3B_a^f\right)\theta_f^d\right\}=$$
$$=-\frac{1}{2}\delta_{ab}^{cd}dc.$$

Since, $c\in C^\infty(M)$ dc (where, $d(\pi^*c)=\pi^*(dc)$) is a horizontal form, and $dc=c_a\omega^a+c^a\omega_a$ therefore. Then (3) can be written as

$$-B^{fdh}B_{abh}\theta_f^c-B^{cfh}B_{abh}\theta_f^d+B^{cdh}B_{fbh}\theta_a^f+B^{cdh}B_{bfh}\theta_b^f+$$
$$+\frac{1}{4(n-1)}\delta_a^c\left\{A_{bhg}^{dh}\omega^g+A_{bh}^{dhg}\omega_g+\left(A_f^d+3B_f^d\right)\theta_b^f-\left(A_b^f+3B_b^f\right)\theta_f^d\right\}+$$
$$+\frac{1}{4(n-1)}\delta_b^d\left\{A_{ahg}^{ch}\omega^g+A_{ah}^{chg}\omega_g+\left(A_f^c+3B_f^c\right)\theta_a^f-\left(A_a^f+3B_a^f\right)\theta_f^c\right\}-$$
$$-\frac{1}{4(n-1)}\delta_a^d\left\{A_{bhg}^{ch}\omega^g+A_{bh}^{chg}\omega_g+\left(A_f^c+3B_f^c\right)\theta_b^f-\left(A_b^f+3B_b^f\right)\theta_f^c\right\}-$$
$$-\frac{1}{4(n-1)}\delta_b^c\left\{A_{ahg}^{dh}\omega^g+A_{ah}^{dhg}\omega_g+\left(A_f^d+3B_f^d\right)\theta_a^f-\left(A_a^f+3B_a^f\right)\theta_f^d\right\}=$$
$$=-\frac{1}{2}\delta_{ab}^{cd}\left(c_g\omega^g+c^g\omega_g\right).$$

The last equality can be rewritten as:

$$-\left[B^{fdh}B_{abh}+\frac{1}{4(n-1)}\left\{\delta_b^d\left(A_a^f+3B_a^f\right)+\delta_a^f\left(A_b^d+3B_b^d\right)-\delta_a^d\left(A_b^f+3B_b^f\right)-\delta_b^f\left(A_a^d+3B_a^d\right)\right\}\right]\theta_f^c-$$
$$-\left[B^{cfh}B_{abh}+\frac{1}{4(n-1)}\left\{\delta_a^c\left(A_b^f+3B_b^f\right)+\delta_b^f\left(A_a^c+3B_a^c\right)-\delta_b^c\left(A_a^f+3B_a^f\right)-\delta_a^f\left(A_b^c+3B_b^c\right)\right\}\right]\theta_f^d+$$
$$+\left[B^{cdh}B_{fbh}+\frac{1}{4(n-1)}\left\{\delta_b^d\left(A_f^c+3B_f^c\right)+\delta_f^c\left(A_b^d+3B_b^d\right)-\delta_f^d\left(A_b^c+3B_b^c\right)-\delta_b^c\left(A_f^d+3B_f^d\right)\right\}\right]\theta_a^f+$$
$$+\left[B^{cdh}B_{afh}+\frac{1}{4(n-1)}\left\{\delta_a^c\left(A_f^d+3B_f^d\right)+\delta_f^d\left(A_a^c+3B_a^c\right)-\delta_f^c\left(A_a^d+3B_a^d\right)-\delta_a^d\left(A_f^c+3B_f^c\right)\right\}\right]\theta_b^f+$$
$$+\frac{1}{4(n-1)}\left\{\left[\delta_a^f\left(A_b^d+3B_b^d\right)-\delta_b^f\left(A_a^d+3B_a^d\right)\right]\theta_f^c+\left[\delta_b^f\left(A_a^c+3B_a^c\right)-\delta_a^f\left(A_b^c+3B_b^c\right)\right]\theta_f^d\right\}-$$
$$-\frac{1}{4(n-1)}\left\{\left[\delta_f^c\left(A_b^d+3B_b^d\right)-\delta_f^d\left(A_b^c+3B_b^c\right)\right]\theta_a^f-\left[\delta_f^d\left(A_a^c+3B_a^c\right)-\delta_f^c\left(A_a^d+3B_a^d\right)\right]\theta_b^f\right\}+$$
$$+\frac{1}{4(n-1)}\left\{\left(\delta_a^cA_{bgh}^{dg}+\delta_b^dA_{agh}^{cg}-\delta_a^dA_{bgh}^{cg}-\delta_b^cA_{agh}^{dg}\right)\omega^h+\left(\delta_a^cA_{bg}^{dgh}+\delta_b^dA_{ag}^{cgh}-\delta_a^dA_{bg}^{cgh}-\delta_b^cA_{ag}^{dgh}\right)\omega_h\right\}=$$
$$=-\frac{1}{2}\delta_{ab}^{cd}\left(c_h\omega^h+c^h\omega_h\right)$$

which in view of (1) takes the form

$$\frac{c}{2}\delta_{ab}^{fd}\theta_f^c+\frac{c}{2}\delta_{ab}^{cf}\theta_f^d-\frac{c}{2}\delta_{fb}^{cd}\theta_a^f-\frac{c}{2}\delta_{af}^{cd}\theta_b^f+\frac{1}{4(n-1)}\left\{\left(A_b^d+3B_b^d\right)\theta_a^c-\left(A_a^d+3B_a^d\right)\theta_b^c\right\}+$$
$$+\frac{1}{4(n-1)}\left\{\left(A_a^c+3B_a^c\right)\theta_b^d-\left(A_b^c+3B_b^c\right)\theta_a^d\right\}-$$
$$-\frac{1}{4(n-1)}\left\{\left(A_b^d+3B_b^d\right)\theta_a^c-\left(A_b^c+3B_b^c\right)\theta_a^d-\left(A_a^c+3B_a^c\right)\theta_b^d+\left(A_a^d+3B_a^d\right)\theta_b^c\right\}+$$
$$+\frac{1}{4(n-1)}\left\{\left(\delta_a^cA_{bgh}^{dg}+\delta_b^dA_{agh}^{cg}-\delta_a^dA_{bgh}^{cg}-\delta_b^cA_{agh}^{dg}\right)\omega^h+\left(\delta_a^cA_{bg}^{dgh}+\delta_b^dA_{ag}^{cgh}-\delta_a^dA_{bg}^{cgh}-\delta_b^cA_{ag}^{dgh}\right)\omega_h\right\}=$$
$$=-\frac{1}{2}\delta_{ab}^{cd}\left(c_h\omega^h+c^h\omega_h\right)$$

and we have

1) $\frac{1}{2(n-1)}\left(\delta_a^dA_{bgh}^{cg}+\delta_b^cA_{agh}^{dg}-\delta_a^cA_{bgh}^{dg}-\delta_b^dA_{agh}^{cg}\right)=\delta_{ab}^{cd}c_h$;

2) $\frac{1}{2(n-1)}\left(\delta_a^dA_{bg}^{cgh}+\delta_b^cA_{ag}^{dgh}-\delta_a^cA_{bg}^{dgh}-\delta_b^dA_{ag}^{cgh}\right)=\delta_{ab}^{cd}c^h$. … (4)

From (4) we have

1) $\frac{1}{2(n-1)}\left(\delta_a^dA_{bgh}^{cg}+\delta_b^cA_{agh}^{dg}-\delta_a^cA_{bgh}^{dg}-\delta_b^dA_{agh}^{cg}\right)=\delta_{ab}^{cd}c_h$;

2) $\frac{1}{2(n-1)}\left(\delta_a^dA_{bg}^{cgh}+\delta_b^cA_{ag}^{dgh}-\delta_a^cA_{bg}^{dgh}-\delta_b^dA_{ag}^{cgh}\right)=\delta_{ab}^{cd}c^h$.

Rolling recent last equality in the first by indices a and c, and then the indices b and d, we obtain:

$$1)\ c_h = \frac{1}{n(n-1)} A^{ab}_{abh}; \quad 2)\ c^h = \frac{1}{n(n-1)} A^{abh}_{ab}. \quad (5)$$

Rolling equation (4), first in the indices a and d, and then the indices b and c, we get:

$$1)\ c_h = -\frac{1}{n(n-1)} A^{ab}_{abh}; \quad 2)\ c^h = -\frac{1}{n(n-1)} A^{abh}_{ab}. \quad (6)$$

Comparing (5) and (6) we find that: $c_h = c^h = 0$.

This theorem proved.

Theorem 3. Nearly Kahler manifold conharmonic constant type is a manifold of constant scalar curvature.

The proof:

In particular, from (6) it follows that. From $A^{ab}_{abh} = A^{abh}_{ab} = 0$ this and from (2: 1) it follows that $dA = dA^a_a = dA^{ab}_{ab} = A^{ab}_{abc}\omega^c + A^{abc}_{ab}\omega_c + A^a_c\theta^c_a - A^c_a\theta^a_c = 0$.

5

From (2: 3), we have $dB = dB^a_a = B^a_c\theta^c_a - B^c_a\theta^a_c = 0$. So from the last two equations, we have that $d\chi = 2dA + 6dB = 0$, that $\chi = const$.

This theorem proved.

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نوع الثابت العالمي الكونهورمني لمنطوي كوهلر التقريبي

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الملخص

الثابت الكونهورمني لمنطوي كوهلر التقريبي والذي يتحقق عندما منطوي كوهلر التقريبي يكون منطوي كونهورمني ذات ثابت معين. حيث تم برهان الثابت المحلي الكونهورمني من نوع المنطوي كوهلر التقريبي يكون مساوي الى نوع الثابت المحلي الكونهورمني. وبرهن ايضا كل منطوي كوهلر التقريبي ذات ثابت نوع كونهورمني هو منطوي ذات ثابت منحني عددي