Regional Exponential Observation and error

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Abstract:
The purpose of this paper is to provide original results related to the choice of the number of sensors and their supports for distributed parameter systems. We introduce the notion of exponential observation error. We show that, the number and location of sensor may be some interest in the existence of regional exponential observation state.

Keywords: regional strategic sensor, exponential detectability, exponential observation.

1. Introduction

In modern mathematical control theory, observability means that it is possible to reconstruct uniquely the initial state of the dynamic system from the knowledge of the input and output [1][15]. Notion of regional observability (extended by El Jai *et al.* [4-5]) is of great importance in current research and motivated by many applications [9][16]. The concept of regional asymptotic analysis was explored by Al-Saphory and El Jai [11] [14], and this concept consist to study the behavior of the system not in whole domain Ω but only on particular region ω of the domain Ω. The purpose of this paper, is to bring the light to link between the exponential observability and sensor structure (see figure 1), we introduce the notion of exponential observation error. We consider a class of distributed system and develop different results connected with the various types of measurements, and we define anew type of strategic sensor which maybe regional exponential strategic sensor.

This paper is organized as follows: section one is focused on preliminaries and the problem formulation. In the next section, the characterization notion of regional exponential observability is given by using of strategic sensors. The last section is devoted to applications with many situations of sensor locations.

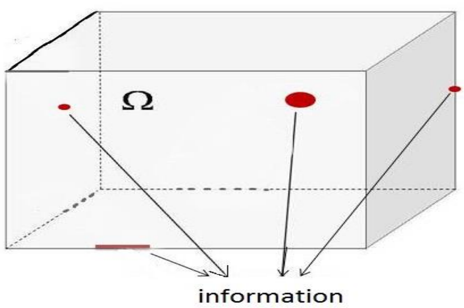

Figure1. The domain Ω and various types of sensors.

2. Preliminaries

In this section, the conceded system and its hypothesis will be given, and the concept of observability will discussed and analyzed in connection with the sensors structures (the locations and numbers of sensors).

2.1 System Definition [7]:

Consider a class of linear distributed parameter systems where the dynamics can be described by the given state equation

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad 0 < t < T \\ z(0) = z_0 \end{cases} \qquad (2.1)$$

where the state space is a Hilbert space and given as $Z = L^2(\Omega)$, and the set Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The domain Ω stands for the geometrical support of the system defined by (2.1). The operator A is a linear operator describing the dynamics of system (2.1), and its generate a strongly continuous semigroup $((S(t))_{t\geq 0}$ on Z. The operator $B \in L(U, Z)$ is the input operator, and $u \in L^2((0,T);U)$, the space U is a Hilbert control space. The considered system is augmented by the output equation

$$y(t) = Cz(t), \qquad (2.2)$$

where $C \in L(Z, \mathcal{O})$, such that $\mathcal{O} = L^2(0,T;Y)$ is a Hilbert observation space.

2.2 Observability notion

For distributed parameter systems, observability can be studied on the autonomous system associated with (2.1)[13], that is:

$$\begin{cases} \dot{z}(t) = Az(t), \quad 0 < t < T \\ z(0) = z_0 \end{cases} \qquad (2.3)$$

so that using (2.2) and (2.3), we obtain the following equalities:

$$y(t) = CS(t)z_0 = Kz_0 \qquad (2.4)$$

Where

$$K : Z \to \mathcal{O} \qquad (2.5)$$

is linear bounded operator, and therefore

$$K^* \in L(\mathcal{O}, Z) \qquad (2.6)$$

is given by the following expression:

$$K^*z = \int_0^T S^*(t)C^*z(t)dt \qquad (2.7)$$

<u>Definition 2.3</u>: [2]

The system (2.1)-(2.2) is said to be weakly observable on [0, T] if the following condition fulfilled:

$$\text{Ker}\{K\} = \{0\} \qquad (2.8)$$

Such that

$$\text{Ker}\{K\} = \{z \in Z / Kz = 0\} \qquad (2.9)$$

Definition 2.4:[2]
The system (2.1)-(2.2) is said to be exactly observable on $[0,T]$ if the following condition satisfied:

$$\|CS(.)z_0\|_{L^2[0,T;Y]} \geq \gamma\|z_0\|_Z,\ \gamma > 0 \qquad (2.10)$$

2.5 Observability and sensors

The sensors have an important role in a System theory. There are give information (measurements) about the state of the system. Sensors can be of different types such that maybe are zonal or pointwise, internal or boundary.
In the following, we give the mathematical definitions for sensors.

Definition 2.6:[8]
Let $D_i \subseteq \Omega$ be a closed and $f_i \in L^2(D_i)$, a sensor is a couple $(D_i, f_i)_{1\leq i\leq q}$ where D_i is the support of the sensor and f_i is the spatial distribution of the sensor.
Some sensors have structures such that they allow the system to be observable or not. In case, when the measurements of system (2.1) are given by ***ith*** sensors$(1 \leq i \leq q)$, then the output function (2.2) is given by

$$y(.,t) = y_1(.,t), \ldots, y_q(.,t) \qquad (2.11)$$

with

$$y_i(.,t) = z(b_i, t), b_i \in \bar{\Omega} \text{ for } 1 \leq i \leq q \qquad (2.12)$$

in the pointwise case, and we have

$$y_i(\mu,t) = \int_{D_i} z(\mu,t) f_i(\mu) d\mu, D_i \subset \bar{\Omega} \text{ for } 1 \leq i \leq q \qquad (2.13)$$

in the zonal case (see figure 1).
Now, we recall the definition of strategic sensor.

Definition 2.7 [13]:
A sensor $(D_i, f_i)_{1\leq i\leq q}$ is said to be strategic if the conceder system is weakly observable.

2.8 Regional observability

An important notion of the framework of a DPS is the region. It is generally defined as a sub domain of Ω in which we are especially interested. Instead of considering a problem on the whole domain Ω, it is possible to consider only a subregion ω of Ω. This has allowed the generalization of the concepts, theorems, and results of the analysis of DPSs to any subdomain of Ω [6].
This section is concerned with the notion of regional observability such that we explained it by the definitions, examples and theorems.
Firstly, we assume that z is the state of a linear system with state space $Z = L^2(\Omega)$, and suppose that the initial state z_0 is unknown. The problem to be studied here concerns the reconstruction of the initial state z_0 on the subregion ω.
Let Ω be a regular bounded open subset of $\mathbb{R}^n$ with boundary ∂Ω, ω be a nonempty subset of Ω, and $[0, T]$ with $T>0$ be a time interval. We denote $Q = \Omega \times (0, \infty)$ and $\Theta = \partial\Omega \times (0, \infty)$, and we consider the autonomous system described by the state equation;

$$\begin{cases} \frac{\partial z}{\partial t}(\mu,t) = Az(\mu,t) & Q \\ z(\eta,t) = 0 & \Theta \\ z(\mu,0) = z_0(\mu) & \Omega \end{cases} \qquad (2.14)$$

And the measurements are given by the output function

$$y(t) = Cz(t), \qquad (2.15)$$

Now, we consider the following decomposition:

$$z_0 = \begin{cases} z_0^e \text{ if } z \in \omega \\ z_0^u \quad \text{if } z \in \Omega / \omega, \end{cases} \qquad (2.16)$$

where z_0^e is the state to be estimated, and z_0^u is the undesired part of the state. The problem of regional observability is consisting in reconstruct the state z_0^e with the knowledge of (2.14) and (2.15).
Consider now, the restriction mapping

$$\chi_\omega : L^2(\Omega) \rightarrow L^2(\omega) \qquad (2.17)$$

defined by

$$\chi_\omega z = z|_\omega \qquad (2.18)$$

where $z|_\omega$ is the restriction of z to ω.

Definition 2.9: [3] A system (2.14)–(2.15) is said to be exactly regionally observable on ω (or ω-observable) if

$$\mathrm{Im}(\chi_\omega K^*) = L^2(\omega) \qquad (2.19)$$

Lemma 2.10: [13] A system (2.14)–(2.15) is exactly ω-observable if there exists $\vartheta>0$ such that, for every $z_0 \in L^2(\omega)$,

$$\|\chi_\omega z_0\|_{L^2(\omega)} \leq \vartheta \|K\chi_\omega{}^* z_0\|_{L^2[0,T;Y]} \qquad (2.20)$$

Definition 2.11: [14]
A system (2.14)–(2.15) is said to be weakly regionally observable on ω (or weakly ω-observable) if

$$\overline{\mathrm{Im}(\chi_\omega K^*)} = L^2(\omega) \qquad (2.21)$$

Remark 2.12:[3] If the system is observable, then it is ω- observable for any $\omega \subset \Omega$.

2.13 Regional observability and sensors

Some sensors have structures such that they allow the system to be observable or regionally observable (figure.2).

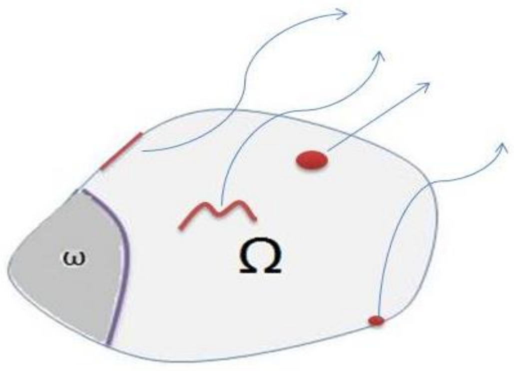

Figure .2 The domain Ω, the region ω, and different types of sensors structures.

Definition 2.14:[11] A sensor $(D_i, f_i)_{1\leq i\leq q}$ is said to be ω- strategic if the conceder system is weakly ω-observable.

Remark 2.15: If the sensor is Ω-strategic, then it is ω-strategic.

Proof:

Assume that the sensor (*D, f*) is Ω-strategic sensor for the conceder system, that is the corresponding system is weakly observable on Ω [13], this lead to the conceder system is weakly ω-observable (the observability implies to regional observability (remark 2.8)) [3], and then the sensor (*D, f*) is ω-strategic.

Now, the following proposition gives the guarantee to the sensor to be ω- strategic.

Proposition 2.16:[14]

The suite of sensors $(D_i, f_i)_{1\le i\le q}$ *is ω- strategic if and only if;*

(1)$q \ge r$,

(2) $rank\ G_n = r_n$ for all n, $n = 1, \ldots, J$

Where

$$G_n = (G_n)_{ij} = \begin{cases} \langle \varphi_{nj}, f_i(.)\rangle_{L^2(D_i)}, & \text{in the zone case} \\ \varphi_{nj}(b_i), & \text{in the pointwise case} \end{cases} \qquad (2.22)$$

when $\sup r_n = r$.

2.17 Regional observability state reconstruction

The purpose of this subsection is to give the estimation of the original state of the system (2.14) on the subregion of Ω [6], that is to give an approach to reconstruct the state z_0^e(equation 2.16).

We consider the autonomous system associated with (2.14) with the hypothesis in section.2, and the decomposition of the initial state (2.16),

with the out put

$$y(t) = Cz(t) \qquad (2.23)$$

Let G and $\bar{G}$ be the sets defined by

$$G = \{\mathrm{g} \in L^2(\Omega) |\ \mathrm{g} = 0\ in\ \Omega \setminus \omega\omega\} \qquad (2.24)$$

and

$$\bar{G} = \{\bar{\mathrm{g}} \in L^2(\Omega) |\ \bar{\mathrm{g}} = 0\ in\ \omega\omega\} \qquad (2.25)$$

then $\forall(\mathrm{g}, \bar{\mathrm{g}}) \in G \times \bar{G}$, we have

$$(\mathrm{g}, \bar{\mathrm{g}}) = \int_\Omega \mathrm{g}\bar{\mathrm{g}}\, dx = \int_{\omega\omega\omega} \mathrm{g}\bar{\mathrm{g}}\, dx + \int_{\Omega\setminus\omega\omega} \mathrm{g}\bar{\mathrm{g}}\, dx = 0 \qquad (2.26)$$

for a given $\varphi_0 \in G$, the system

$$\begin{cases} \dfrac{\partial\varphi}{\partial t}(\mu, t) = A\varphi(\mu, t) & Q \\ \varphi(\mu, 0) = \varphi_0(\mu) & \Omega \qquad (2.27) \\ \varphi(\eta, t) = 0 & \Theta \end{cases}$$

has a unique solution φ. The mapping

$$\varphi_0 \in G \ \rightarrow \|\varphi_0\|_G^2 = \int_0^T \varphi^2(b, t) dt \qquad (2.28)$$

defines a semi-norm on G. and $b \in \Omega$ denotes the given location of the sensor (b, δ_b), for $\varphi_0 \in G$, the equation (3.28) gives φ which allows to consider the system

$$\begin{cases} \frac{\partial\psi_1}{\partial t}(\mu, t) = -A^*\psi_1(\mu, t) - \varphi(b, t)\delta(\mu - b) & Q \\ \psi_1(\mu, T) = 0 & \Omega \qquad (2.29) \\ \psi_1(\eta, t) = 0 & \Theta \end{cases}$$

let ψ_1 be the solution of (2.29). consider the operator $\wedge$ defined by

$$\wedge\colon\ G \ \rightarrow \ G^\perp \qquad (2.30)$$

$$\varphi_0 \rightarrow \ P_{G^\perp}(\psi_1(0))$$

where $P_{G^\perp}(\psi_1(0))$ denotes the projection of $\psi_1(0)$ on $G^\perp$.

Now, consider the system

$$\begin{cases} \frac{\partial\psi_2}{\partial t}(\mu, t) = -A^*\psi_2(\mu, t) - y(b, t)\delta(\mu - b) & Q \\ \psi_2(\mu, T) = 0 & \Omega \qquad (2.31) \\ \psi_2(\eta, t) = 0 & \Theta \end{cases}$$

if φ_0 is such that φ leds to $\psi_1(0) = \psi_2(0)$ on (2.31) ω then the system (2.32) looks like the adjoint of the system to be observed (2.14)-(2.15) and consequently, the observation problem on ωω is equivalent to solve the equation

$$\wedge\varphi_0 = P_{G^\perp}(\psi_2(0)) \qquad (2.32)$$

Proposition 2.18: [8]

If the sensor (b, δ_b) is ω-strategic, then (2.32) has a unique solution $\varphi_0 \in G$ which corresponds to the regional state z_0^e to be observed in ω.

2.19 Regional observability and error

When the system is observable, the state reconstruction leads necessarily to a reconstruction error (figure.3), also called the observation error.

This subsection concerned to the observation error, in the usual observation problem, we consider the error $E(z_0)$ defined by

$$E(z_0) = \|y(t) - \tilde{y}(t)\|_{L^2[0,T;Y]} \qquad (2.33)$$

where ỹ holds for the measurement and y for the output. In our case, the observation error clearly depends on the target region ω where the state is to be observed together with the structure and number of sensors.

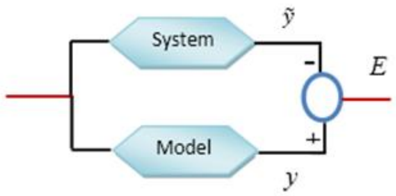

Figure 3.the observation error

2.20 Regional exponential observability

In this section the concepts of exponential stable, exponential detectable and exponential observable are explained and analyzed also an approach which observes the state on ω exponentially will be avoids.

2.21 Regional exponential detectability

The concept of regional exponential observability needs some notions which are related to the

exponential behavior, which is stability, detectability and observer. The exponential behavior nation has been extended recently by Al-Saphory and El Jai [10].
We have the following definitions and propositions as stated in [8-10].
Definition 2.22: A semigroup $(S_A(t))_{t\geq 0}$ is said to be exponentially regionally stable on ω (or ω_E- stable) if there exist appositive constants M_ω and α_ω such that:

$$\|\chi_\omega S_A(t)\|_{L^2(\omega)} \leq M_\omega e^{-\alpha_\omega t}, t\geq 0 \quad (2.34)$$

If $(S_A(t))_{t\geq 0}$ is ω_E-stable, then for all $z_0(t) \in L^2(\Omega)$, the solution of autonomous system associated with (2.14) satisfies

$$\|z(t)\|_{L^2(\omega)} = \|\chi_\omega S_A(t) z_0\|_{L^2(\omega)} \leq M_\omega e^{-\alpha_\omega t}\|z_0\| \quad (2.35)$$

and then

$$lim_{t\to\infty}\|z(t)\|_{L^2(\omega)} = lim_{t\to\infty}\|\chi_\omega S_A(t) z_0\|_{L^2(\omega)} = 0 \quad (2.36)$$

Definition 2.23:[14] The system (2.14)-(2.15) is said to be exponentially regionally stable on ω (or ω_E-stable) if the operator A generates a semigroup which is ω_E-stable.
If it possible to detect exponentially the current state of the original system in a given sub region ω of Ω, then this reconstruction is regional exponential detectability.
Definition 2.24:[10] The system (2.14) together with output (2.15) is said to be exponentially regionally detectable on ω (or ω_E-detectable) if there exists an operator $H_\omega : L^2(0,T,R^q) \to L^2(\omega)$ such that $(A-H_\omega C)$ generates a strongly continuous semigroup $(S_{H_\omega}(t))_{t\geq 0}$ which is ω_E-stable.
Proposition 2.25: If the system (2.14)-(2.15) is regionally exactly observable on ω , then it is regionally exponentially detectable on ω.
Proof:
We have if the system is exactly observable then it is exponentially detectable [13], this is given by the relation;

$$\exists\gamma>0, \|z\|_Z \leq \gamma\|CS_A(.)z\|_O, \forall z\in L^2(\Omega) \quad (2.37)$$

Now, since the observability implies to regional observability [15] and detectability lead to regional exponential detectability [7], thus if the system (2.14)-(2.15) is regional exactly observable on ω then it is regional exponential detectable on ω, and this is given by the equality;

$$\exists\gamma>0, \|\chi_\omega S_A(.)z\|_{L^2(\omega)} \leq \gamma\|CS_A(.)z\|_O, \forall z\in L^2(\omega). \quad (2.38)$$

Here, the following proposition explains the relation between regional exponential detectability and sensor structure. For that purpose, let us consider the set $(\varphi_i)$of functions of $L^2(\Omega)$ orthonormal in $L^2(\omega w)$ associated with the eigenvalues λ_i of multiplicity m_i and suppose that the system (2.14) has J unstable modes.
Proposition 2.26: Suppose that there are q sensors $(D_i,f_i)_{1\leq i\leq q}$ and the spectrum of A contains J eigenvalues with nonnegative real parts. Then the system (2.14)-(2.15) is ω_E-detectable if and only if:
(1)$q\geq m$,
(2) $rank\, G_i = m_i$ for all i, $i = 1, \ldots, J$ with

$$G=(G_{ij}) = \begin{cases} \langle\varphi_j(.), f_i(.)\rangle_{L^2(D_i)}, & \text{for zone sensors} \\ \varphi_j(b_i), & \text{for pointwise sensors} \\ \langle\varphi_j(.), f_i(.)\rangle_{L^2(\Gamma_i)}, & \text{for boundary zone sensors} \end{cases} \quad (2.39)$$

where $\sup m_i = m < \infty$ and $j = 1, \ldots, \infty$.

2.27 Regional exponential observability

The regional exponential observation problem consist of reconstruct the state of the original system exponentially not on whole domain **Ω** but only on consider subregion ω of **Ω**.
Now, consider the system (2.14)-(2.15) together with the dynamical system

$$\begin{cases} \frac{\partial w}{\partial t}(\mu,t) = F_\omega w(\mu,t) + G_\omega u(t) + H_\omega y(t) & Q \\ w(\eta,t) = 0 & \Theta \\ w(\mu,0) = w_0(\mu) & \Omega \end{cases} \quad (2.40)$$

where F_ω generates a strongly continuous semigroup $(S_{F_w}(t))t\geq 0$ which is stable on Hilbert space W ,$G_w \in L(R^p, W)$ and $H_w \in L(W, R^q)$. The system (2.40) defines an ω_E-estimator for $\chi_\omega Tz(\mu,t)$if
1- $\lim_{t\to\infty}\|w(.,t) - \chi_\omega Tz(\mu,t)\|_{L^2(\omega)} = 0 \quad (2.41)$
2- $\chi_\omega T$ maps $D(A)$ into $D(F_{\omega_w})$ where $w(\mu,t)$ is the solution of the system(2.40).
Definition 2.28 [11]:
The system (2.40) specifies a regional exponential observer $(\boldsymbol{\omega_E - observer})$ for the system (2.14)-(2.15) if the following conditions hold:
1- there exist $M_\omega \in L(R^q, L^2(\omega))$ and $N_\omega \in L(L^2(\omega))$ such that

$$M_\omega C + N_\omega \chi_\omega T = I_\omega, \quad (2.42)$$

2- $\chi_\omega TA + F_\omega \chi_\omega T = H_\omega C$ and $G_\omega = \chi_\omega TB$,
3- The system (2.40) defines an ω_E-estimator for the state of the system (2.14).
In the following, we define a new type of strategic sensor which make the system (2.40) form an ω_E-observer for (2.14)-(2.15).
Definition 2.29: The system (2.41) is said to be identity ω_E-observer for the system (2.14)-(2.15) if Z = W and $\chi_\omega T = I_\omega$. In this case, we have $F_\omega = A - H_\omega C$ and $G_\omega = B$. Then, the dynamical system (2.41) becomes:

$$\begin{cases} \frac{\partial w}{\partial t}(\mu,t) = Aw(\mu,t) + Bu(t) + H_{\omega_w}(Cw(\mu,t) - y(.,t)) & Q \\ w(\eta,t) = 0 & \Theta \\ w(\mu,0) = 0 & \Omega \end{cases} \qquad (2.43)$$

Definition 2.30: The system (2.14)-(2.15) is ω_E-observable if there exist a dynamical system (2.41) which is ω_E-observer for the original system.

Now, we define the suite of sensor which make the system (2.14)-(2.15) is ω_E-observable.

Definition 2.31: The suite of sensor $(D_i, f_i)_{1\le i\le q}$ is said to be regional exponential strategic (or ω_E-strategic) if the observed system is ω_E-observable.

In the following, an example for the system which is not observable on Ω, but it is ω_E-observable will be given.

Example 2.32 : Consider the system

$$\begin{cases} \frac{\partial z}{\partial t}(\mu,t) = \Delta z(\mu,t) + z(\mu,t) & Q \\ z(\eta,t) = 0 & \Theta \\ z(\mu,0) = 0 & \Omega \end{cases} \qquad (2.44)$$

augmented

with the output function

$$y(t) = \int_\Omega z(\mu,t)\delta(\mu - b_i)\, d\mu \qquad (2.45)$$

Where Ω = (0, 1) and $b_i \in \Omega$ are the location of sensors $(b_i,\ \delta b_i)$ as in (Figure 5). The operator $(\Delta + 1)$ generates a strongly continuous semigroup $(S_\Delta(t))_{t\ge 0}$ on the Hilbert space $L^2(\omega)$.

Consider the dynamical system

$$\begin{cases} \frac{\partial w}{\partial t}(\mu,t) = \Delta w(\mu,t) + w(\mu,t) + H(Cw(\mu,t) - y(.,t)) & Q \\ w(0,t) = w(1,t) = 0 & \Theta \\ w(\mu,0) = w_0(t) & \Omega \end{cases} \qquad (2.46)$$

where $H \in L(R^q, W)$, W is the Hilbert space, and C : $W \to R^q$ is linear operator. If $b_i \in$ Q, then the sensors $(b_i, \boldsymbol{\delta} b_i)$ are not strategic for the unstable subsystem (2.44) [19] and therefore the system (2.44)-(2.45) is not exponentially detectable in Ω [20]. Then, the dynamical system (2.46) is not exponential observer for (2.44)-(2.45).

Figure 4: The domain Ω, the subregion ω, and locations b_i of internal pointwise sensors.

Now, we consider the region $\omega = (0,\beta) \subset (0,1)$and the dynamical system

$$\begin{cases} \frac{\partial w}{\partial t}(\mu,t) = \Delta w(\mu,t) + w(\mu,t) + H_\omega(Cw(\mu,t) - y(.,t)) & Q \\ w(0,t) = w(1,t) = 0 & \Theta \\ w(\mu,0) = w_0(t) & \Omega \end{cases} \qquad (2.47)$$

where $H_\omega \in L(R^q, L^2(\omega))$. If $b_i/\beta \notin Q$, then the sensors $(b_i,\ \delta b_i)$ are ω-strategic for the unstable subsystem of (2.44) [8] and then the system (2.44)-(2.45) is ω_E-detectable [10]. Therefore, the system (2.44)-(2.45) is ω_E-observable by (2.47).

3. Regional exponential observation and error

We can give the following proposition which show that the importance of the exponential observation in minimizing the error between the original state and estimating state.

The importance of exponential reconstruction for the state of consider system on ω can be seen by study the error between the original state $z(\mu,t)$ of consider system and the estimation state $\tilde{z}(\mu,t)$ of the estimator system which accurse through the reconstruction, this error is given by the equation below:

$$e(\mu,t) = z(\mu,t) - \tilde{z}(\mu,t) \qquad (3.1)$$

This error calls reconstruction error (observation error). This problem of error was treatment in the exponentially reconstruction such that the observation error converges exponentially to zero as $\boldsymbol{t}$ tends to ∞ , this comes from the stable semi-group $(S_A(t))t \ge 0$

Proposition 3.1. The reconstruction error fading by exponential observation.

Proof:

Let $(S_A(t))t \ge 0$ be a semi-group such that :

$\|S_A(t)\|_{L^2(\Omega)} \le Fe^{-\sigma t}, t \ge$ 0 for some bositive constants F and σ, (3.2)

and in region $L^2(\omega)$ when the regional stable semigroup $(S_A(t))t \ge 0$ satisfied:

$$\|\chi_\omega S_A(.)\|_{L^2(\omega)} \le F_\omega e^{-\sigma_\omega t}, t \ge 0,$$

for some positive constants F_Γ and σ_Γ (3.3)

Thus, we have

$$\begin{aligned} \|e(\mu,t)\|_{L^2(\omega)} &= \|z(\mu,t) - w(\mu,t)\|_{L^2(\omega)} \\ &\le \left\|\chi_\omega S_{H_\omega}(t) e_0\right\|_{L^2(\omega)} \\ &\le F_\Gamma e^{-\sigma_\Gamma t}\|e_0\|_{L^2(\omega)} \end{aligned} \qquad (3.4)$$

Consequently, we get

$\lim_{t\to\infty}\|e(\mu,t)\|_{L^2(\omega)} = \lim_{t\to\infty}\|\chi_\omega S_A(t)e_0\|_{L^2(\omega)} = 0$. (3.5)

4. Application to sensor location

In this subsection, we apply the previous results to two dimensional system defined on Disc domain, In this case, we consider the system

$$\begin{cases} \frac{\partial z}{\partial t}(r,\theta,t) = \Delta z(r,\theta,t) + z(r,\theta,t) + Bu(t) & Q \\ z(r,\theta,0) = z_0(r,\theta) & \Omega \\ z(r,\theta,t) = 0 & \Sigma \end{cases} \qquad \ldots(4.1)$$

where $\Omega = D(0,1)$,$\theta \in [0,2\pi]$,$\omega = D(0,r_\omega) \subset \Omega$, the eigenfunctions and eigenvalues concerning the system (4.1) are given by the Bessel functions as follows

$$\lambda_{ij} = -\beta^2 ij, i \ge 0, j \ge 1 \qquad (4.2)$$

where β_{ij} are the zeros of the Bessel functions J_i

$$\begin{cases} \varphi_{0j}(r,\theta) = J_0(\beta_{0j} r), & j \geq 1 \\ \varphi_{ij_1}(r,\theta) = J_i(\beta_{ij_1} r)\cos(i\theta), & i, j_1 \geq 1 \\ \varphi_{ij_2}(r,\theta) = J_i(\beta_{ij_2} r)\sin(i\theta), & i, j_2 \geq 1 \end{cases} \quad (4.3)$$

and the multiplicity $m_i = 2$ for all $i, j \neq 0$, and $m_i = 1$ for $i, j = 0$.

The following results give information on the position of pointwise or zonal sensors which are ω-strategic.

4.1 Internal Pointwise Sensor

Let us consider the case of pointwise sensor located inside of $\Omega = D(0,1)$. The system (4.1) is augmented with the following output function:

$y(t) = \int_\Omega z(r_i, \theta_i, t)\, f_i(r_i, \theta_i) dr_i d\theta_i, 0 \leq \theta_i \leq 2\pi, 0 \leq r_i \leq \frac{1}{2}$, (4.4) such that the sensors located in $c_1 = (r_1, \theta_1)$ and $c_2 = (r_2, \theta_2) \in \Omega$, (fig. 5).

If there exist $i \in \{1, \ldots, J\}$, such that $\frac{i(\theta_1 - \theta_2)}{\pi} \notin I$, then the sensors c_1 and c_2 may be sufficient for Γ_E-observability, then the dynamical system:

$$\begin{cases} \frac{\partial w}{\partial t}(r,\theta,t) = \Delta w(r,\theta,t) + w(r,\theta,t) + Bu(t) + H_\omega(w(r,\theta,t) - y(t)) & Q \\ w(r,\theta,0) = w_0(r,\theta) & \Omega \\ w(r,\theta,t) = 0 & \Sigma \end{cases} \quad (4.5)$$

Forms an ω_E-observer for (4.1), thus we obtain the following result

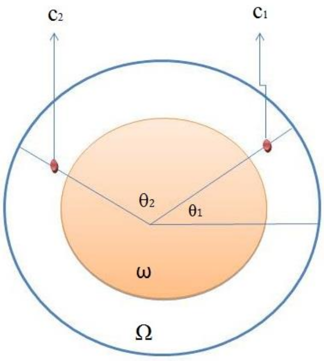

Figure 5: Disc domain Ω, region ω and location c1 and c2 of internal poinwise sensors.

<u>**Corollary 4.2**</u>**:** The system (4.1)-(4.4) is ω_E-observable by the dynamical system (4.5), If $\frac{i(\theta_1 - \theta_2)}{\pi} \notin I$, for $i=1,\ldots,J$.

4.3 Internal Zone Sensor

Consider the system (4.1) together with output function (4.2) where the sensor supports D_1 and D_2 are located in $D(0,1)$. The output (4.2) can be written by the form

$y(t) = \int_{D_i} z(r_i, \theta_i, t)\, f_i(r_i, \theta_i) dr_i d\theta_i, 0 \leq \theta_i \leq 2\pi, 0 \leq r_i \leq \frac{1}{2}$ (4.6)

where $D_1 = (r_1, \theta_1)$ and $D_2 = (r_2, \theta_2) \subset D(0,1)$(see fig. 6) is the location of sensors (D_1, f_1) and (D_2, f_2), if there exist $i \in \{1, \ldots, J\}$ such that f_i are not symmetric with respect to θ_i , and $\frac{i(\theta_1 - \theta_2)}{\pi} \notin I$ and $m_i = 1$, then the sensors (D_1, f_1) and (D_2, f_2) may be sufficient for ω_E-observability, then the dynamical system:

$$\begin{cases} \frac{\partial w}{\partial t}(r,\theta,t) = \Delta w(r,\theta,t) + w(r,\theta,t) + Bu(t) + H_\omega(\langle w(r,\theta,t), f(r,\theta)\rangle - y(t)) & Q \\ w(r,\theta,0) = w_0(r,\theta) & \Omega \\ w(r,\theta,t) = 0 & \Sigma \end{cases} \quad (4.7)$$

forms an ω_E-observer for (2.49), thus we obtain the following result:

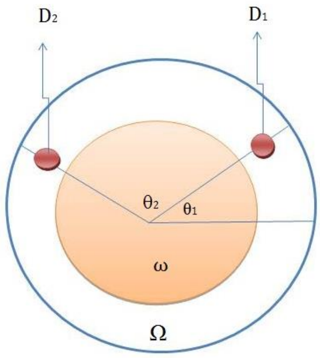

Figure 6: Disc domain Ω, region ω and location *D*1 and *D2* of internal zone sensors.

<u>**Corollary 4.4**</u>**:** The system (4.1)-(4.6) is ω_E-observable by the dynamical system (4.7), If $\frac{i(\theta_1 - \theta_2)}{\pi} \notin I$ and f_i not symmetric with respect to θ_i, for $i=1,\ldots,J$.

4.5 Internal Filament Sensor

Consider the case of the observation on the curve $\sigma = \mathrm{Im}(\gamma)$ with $\gamma \in C^1(0,1)$ (see Fig. 7), then we have the following result;

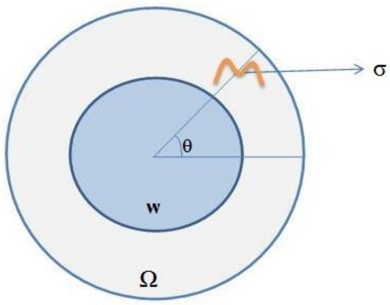

Figure 7: Disc domain, region ω and location *σ* of internal filament sensor.

Corollary 4.6: If the observation recovered by filament sensor $(\sigma, \delta\sigma)$ such that it is symmetric with respect to the curve $\theta = \theta_i$, then the system (4.1) with output given by

$$y(t) = \int_{\Omega} z(r_i, \theta_i, t)\, f_i(r_i, \theta_i) dr_i d\theta_i, \quad (4.8)$$

Is ω_E-observable if there exist $i \in \{1, \ldots, J\}$, such that $i_0\theta \notin I$.

Conclusion

We give in this paper anew characterization of sensor which make the consider system observable in critical region ω. And we show in counter example the system which is not observable on Ω, maybe ω_E-observable. Finally we introduce an proposition shows that the reconstruction error fading by exponential observation.

Many questions still opened. This is the case of, for example, the problem of reconstruction the state of non-linear systems exponentially on a considered sub region and its error.

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المشاهدة الآسية المناطقية والخطأ

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الملخص

الهدف من البحث تزويدنا بنتائج دقيقة تتعلق باختيار عدد المجسات وموقعها للأنظمة التوزيعية. سنقدم في هذا البحث مفهوم الخطأ للمشاهدة الآسية, وسنبين ان عدد ومواقع المجسات يمكن ان يؤثر في المشاهدة الآسية المناطقية للحالة.