Tikrit Journal of Pure Science (2023) 28 (5):158-175
Doi: https://doi.org/10.25130/tjps.v28i5.1587

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المجلات الاكاديمية العراقية

Tikrit Journal of Pure Science
ISSN: 1813 – 1662 (Print) --- E-ISSN: 2415 – 1726 (Online)

Journal Homepage: http://tjps.tu.edu.iq/index.php/j

Regular Divisor Graph of Finite Commutative Ring

Payman Abbas Rashid[1],Hataw Saleem Rashid[2]
[1,2]Department of Mathematics, College of Sciences, Salahuddin University, Kurdistan, Iraq

Keywords: Regular divisor graph, Vertex-connectivity, Edge-connectivity, Block.

ARTICLE INFO.

Article history:

-Received:	09 Jan. 2023
-Received in revised form:	13 Feb. 2023
-Accepted:	14 Feb. 2023
-Final Proofreading:	24 Oct. 2023
-Available online:	25 Oct. 2023

Corresponding Author*:
Payman Abbas Rashid

ABSTRACT

Let R be a finite commutative ring with identity 1. We introduce a new graph called regular divisor graph and denoted by $\mathfrak{R}_{\partial}(R)$. We classify the finite commutative ring to get a special graph and we are going to study some properties of this graph, clique number, chromatic number, number of cycles, connectivity and blocks.

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البيان القسم المنتظم للحلقة التبديلية المنتهية

بيمان عباس رشيد [1] ، هتاو سليم رشيد [2]

[1] قسم الرياضيات، كلية العلوم، جامعة صلاح الدين، إقليم كردستان، العراق

الملخص

ليكن R حلقة تبديلية منتهية، عرفنا بيان جديد يعتمد على الحلقة المنتظمة R سـميت ببيان القاسـم المنتظم يرمز لها $\mathfrak{R}_{\partial}(\mathbf{R})$ لقد قمنا بتصـنيف الحلقة التبديلية R من حيث العناصر المنتظمة لنحصل على بيانات خاصة ومتنوعة، كما درسنا خصائص تلك البيانات من حيث درجة التلوين، عدد الدارات، عدد الرمز وخصائص اخرى متعلقة بالبيان المفصل والحواجز في نظرية البيانات.

Introduction

Let R be a finite commutative ring with identity 1, an element $a \in R$ is called Von Neumann regular if there exist $b \in R$ *such that* $a = a.b.a$, a Ring R is Said to be a Von Neumann regular ring if all elements in R are Von Neumann regular [1], [10]. We denote the set of Von Newman regular elements by $r(R)$ and set of non-zero Von Neumann regular elements by $r^*(R) = r(R) - \{0\}$. Taloukolaei and Sahebi introduced the Von Neumann regular graph $GVnr+(R)$ of a ring R, whose vertex set consists of elements of R and two distinct vertices x and y are adjacent if and only if $x + y$ is a Von Neumann regular element [2]. By taking advantage of their work, we have defined a new graph in this way, let R be a finite commutative ring with 1. That for all non-zero elements a and b in the ring R are adjacent if and only if $a = a.b.a$ or $b = b.a.b$ *and* $a \neq b$, this graph is called by regular divisor graph and denoted by $\mathfrak{R}_{\partial}(R)$. with vertex set $V(\mathfrak{R}_{\partial}(R))$ consists of elements of $r^*(R)$ and edge set $E(\mathfrak{R}_{\partial}(R)) = \{(a,b) : a = a.b.a$ or $b = b.a.b$, $a \neq b \neq 0\}$.
we have used some basic concepts in ring theory from [5],[8],[10]and used some basic concepts in graph theory from [2],[3],[4],[6].

Definition 1.1: [3] A graph G is finite non-empty set consist of two sets, the **set of Vertices** $V(G)$ and the **set of edges** $E(G)$. $V(G)$ is a non-empty set of elements named vertices. While $E(G)$ is the set (which is possible empty) of unordered pairs of vertices of $V(G)$ called **edges**. The **order** of the graph is the number of vertices which is denoted by $\eta(G)$, that is $\eta(G) = |V(G)|$, and the number of edges of G is called the **Size of** G and is denoted by $\Upsilon(G)$, that is $\Upsilon(G) = |E(G)|$.

Definition 1.2: [4] The **degree** of a vertex v of a graph G is the number of all edge's incident to v in G. We denote the degree of the vertex v of G by $deg(v)$. The **Center** of a graph G is the vertex v which has greatest degree.

Definition 1.3: [3] A **Walk** W in G is an edge, starting at v_1 and ending at v_j such that consecutive vertices in W are adjacent. A walk in which no vertex is repeated is called a **Path.** A path with n vertices is denoted by P_n . A path that begins and ends at the same vertex is called circuit.
A cycle with $n \geq 3$ vertices is denoted by C_n .

Definition 1.4 Let G be a connected graph, the **eccentricity** of vertex $v \in V(G)$, denoted by $e(v)$ is the distance between v and a vertex furthest from v. The diameter of G is the maximum distance between the pair of vertices, and denoted by $\mathcal{Dim}(G)$. While the radius of G denoted by $\boldsymbol{rad}(\boldsymbol{G})$ is the minimum distance between the pair of vertices.

Definition 1.5 A complete subgraph of a graph G is called **clique of** G**.** And the maximum order of a clique of G is called **clique number od** G and denoted by $\boldsymbol{\omega}(\boldsymbol{G})$. The **girth** of graph G is the size of the smallest cycle in the graph and denoted by $\boldsymbol{gi}(\boldsymbol{G})$.

Definition 1.6: [5] The **chromatic number** of a graph G is denoted by $\boldsymbol{\chi}(\boldsymbol{G})$. Is the minimum number of colors needed for proper vertex coloring of G . G is k-

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chromatic if $\chi(G) = k$. (Where k is positive integer number)

Definition 1.7: Let v_i and v_j be two distinct vertices of graph G_1 and G_2 respectability. Two vertices v_i and v_j are identified if they replaced by anew vertex v^* such that all edges incident on v_i and v_j are now incident on the new vertex v^* and denoted by $\boldsymbol{G_1 \bullet G_2}$.

Definition 1.8 (**Double identifying**) is the identifying two distinct vertices in the graphs G_1 and G_2, denoted by $G_1 \bullet\bullet G_2$, and identifying an edge between two graphs say $e_1 \in G_1$ and $e_2 \in G_2$ is denoted by $G_1 \multimap\!\bullet\ G_2$.

2-Regular divisor graph of commutative ring Z_n

The commutative ring Z_n , $n \geq 1$ for n equal to (prime, composite, odd, or even), is regular ring or not regular ring but in each case, it has some regular elements depending on n.

To study the regular divisor graph of the commutative ring Z_n, which is (undirected) graph and symbolized by $\mathfrak{R}_{\boldsymbol{\partial}}(Z_n)$, where two non-zero distinct elements in Z_n, a and b are adjacent as a vertex if and only if $a = a.b.a$ or $b = b.a.b$ for the regular elements $a, b \in Z_n$. the regular divisor graph of commutative ring Z_n is simple, undirected loop less graph $\mathfrak{R}_{\boldsymbol{\partial}}(Z_n)$ with vertex set $V(Z_n)$ and edge set $E(Z_n) = \{(a,b): a = a.b.a$ or $b = b.a.b$, $a \neq b \neq 0 \in Z_n\}$.

Example 1: The ring Z_{18} which is not regular ring but have some regular elements, the non-zero regular elements $r^*(Z_{18}) =$ {1,24,5,7,8,9,10,11,13,14,16,17} ,the regular divisor graph $\mathfrak{R}_{\boldsymbol{\partial}}(z_{18})$ is shown in figure-2.1, which is different from any other regular divisor graphs.

Gingivitis and periodontitis are two conditions listed under the umbrella term periodontal disease. Periodontal disease refers to a range of conditions that affect the supporting tissues of the teeth [1]. Typically, one of the first indications of gingivitis is bleeding gums, which is a common symptom of the disorder [2]. In the absence of treatment, gingivitis can progress to periodontitis, which is characterized by the loss of periodontal attachment and alveolar bone and ultimately results in tooth loss. Antibiotics can be used to treat gingivitis [3].

Dentists refer to the inflammation of the gums as gingivitis. It occurs as a result of inadequate tooth cleaning, which leads to the deposition of bacterial plaque on the surface of the teeth. Therefore, effective tooth brushing is vital for achieving enough food debris clearance, as it helps to avoid the formation of plaque in the future.

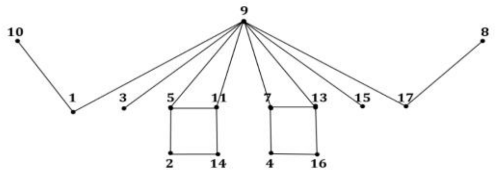

Figure-2.1: Regular divisor graph $\mathfrak{R}_{\boldsymbol{\partial}}(z_{18})$

The regular divisor graph of commutative ring Z_n , $n \geq 1$ has no well-known form (certain form) it is changed with respect on n (prime, composite, odd or even) to find the certain form of the graph we must classify the ring Z_n with respect to the order of ring (n) as the following:

2.1 Regular divisor graph of the ring Z_p for all prime number $p > 3$

The ring Z_p is regular ring for all prime p, since Z_p is a division ring and every division ring is a regular ring. The regular divisor graph of the ring Z_p is special graph, for all non-zero element $a \in Z_p$ there exist $a^{-1} \in Z_p$ such that a and a^{-1} are adjacent.

Theorem 2.1.1 The regular divisor graph of the ring Z_p is bipartite graph and

$\mathfrak{R}_{\boldsymbol{\partial}}(Z_p) \cong \frac{(p-3)}{2} K_2$ where K_2 is a complete graph of order two.

Proof: Since the ring Z_p is division ring then Z_p is regular, it means for all $a \in Z_p$ a is a regular element. Then the vertex set $\mathrm{V}(Z_p) = Z_p - \{0\}$, two elements 1 and

$p-1$ in Z_p are self-regular and self-inverse

$1.1.1 = 1$ and $(p-1)^2.(p-1) = (p-1)$ and they make a loop in the regular divisor graph so we exclude 1 and $p-1$ in the vertex set,

then $|V(Z_p)| = p-3$, and each $a \in Z_p$, a is adjacent with a^{-1} $(a_i.a_i^{-1}.a_i = a_i)$ for all $a_i \in Z_p)$ by the regularity so we have two types of vertices such that $V_1(Z_p) = \{a_1, a_2, \dots, a_i\}$ and $V_2(Z_p) = \{a_1^{-1}, a_2^{-1}, \dots, a_i^{-1}\}$, $i = \frac{p-3}{2}$, and each element in V_1 is adjacent with only one element in V_2, since the inverse element is unique. There is no another adjacent vertex in the graph.

And we have exactly $\frac{p-3}{2}$ vertices in both sets. Then the regular divisor graph of the ring Z_p is bipartite graph and have exactly $\frac{p-3}{2}$ vertices in each partite set and $\Re_\partial(Z_p) \cong \frac{(p-3)}{2} K_2$.

Example 3: Consider the ring Z_{11} , $p = 11$

The non-zero regular elements of Z_{11} are $r^*(Z_{11}) = \{1,2,3,4,5,6,7,8,9,10\}$ the elements 1 and 10 are self-regular then exclude them in the vertex set. And the vertex set is $V(Z_{11}) = \{2,3,4,5,6,7,8,9\}$. The regular divisor graph of z_{11} is shown in figure-2.2-

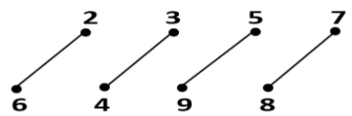

Figure 2.2: Regular divisor graph $\Re_\partial(z_{11})$

Note that the vertex set of the regular divisor graph $\Re_\partial(z_{11})$ contains two types of vertices $V_1 = \{2,3,5,7\}$ *and* $V_2 = \{6,4,9,8\}$ each vertex in V_1 is adjacent with vertex in V_2 respectability since $a_i.b_i.a_i = a_i$ for all $a_i.b_i$ in V_1 and V_2 respectability where $b_i = a_i^{-1}$ then we get only K_2 from each a_i and b_i since we have 4 vertices in V_1 and V_2 then we get 4 copies of K_2 then our graph $\Re_\partial(z_{11}) \cong 4K_2$ and it is a bipartite graph.

2.2 Regular divisor graph of the ring Z_n , $n = qp$ $(q, p$ *are prime number and* $q < p)$

In this case for $n = q.p$, the regular divisor graph has a special shape and different properties. To get the graph of the ring $Z_{q.p}$ we first gave the following example.

Example 4: Consider the ring Z_{10} , where $q = 2$, $p = 5$

Z_{10} is regular ring and we get the graph shown in the figure-2.3-

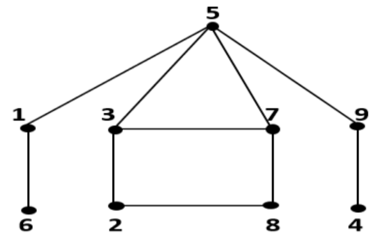

Figure2.3: Regular divisor graph$\Re_\partial(z_{10})$

Some elements in the set of regular elements $r^*(Z_{10})$,$\{1,4,5,6,9\}$ they make a loop in the graph, then we exclude them self-regularity in the vertex set because our graph is simple.

The properties of the graph $\Re_\partial(Z_{10})$

1) P has the maximum degree $deg(p) = p-1$ it is the center of graph
2) Two vertices $(p-1) = 4$ and $(p+1) = 6$ are end vertices
3) Contains one cycle of length 4
4) The degree sequence is $\{4,3,3,2,2,2,2,1,1\}$
5) The $gi(\Re_\partial(Z_{10}))$ $(= \omega(\Re_\partial(Z_{10})) = 3$
6) $Dim(\Re_\partial(Z_{10})) = 2$
7) $p^2 = p$ and $(p+1)^2 = p+1$ are idempotent elements.

2.3 Regular divisor graph of the ring Z_{2p} for all prime number $p > 2$

In general, the regular divisor graph of the ring Z_{2p} , $\Re_\partial(Z_{2p})$ has a vertex set $V(Z_{2p})$ of non-zero distinct regular elements, since the ring of Z_{2p} is always regular, then the vertex set $V(Z_{2p})$ is non-empty set, the edge set $E(Z_{2p})$ is the set of edges ab, where a and b are adjacent with respect to regularity. We divide the vertex set in to two partite sets with respect to the adjacency of the vertices as a regular element in the regular ring Z_{2p} .

$V(Z_{2p}) = V_1(Z_{2p}) \cup V_2(Z_{2p})$, where V_1 and V_2 are two partite sets of $V(Z_{2p})$ such that

$V_1(Z_{2p}) = \{1,3,5,\dots,p-2,p,p+2,\dots,2p-1\} = odd$ elements, contains all odd regular elements where they are unit elements except p (where p is idempotent element in Z_{2p}).

$V_2(Z_{2p}) = \{2,4,6,\dots,2(p-1)\}$, contains even regular elements.

The first partite set of vertices V_1 has exactly p vertices and the second partite set V_2 has exactly $(p-1)$ vertices. Two vertices $(p-1)$ and $(p+1)$ are end vertices where they are adjacent with two regular elements $2p-1$ and 1 respectability. When this last two vertices are adjacent with p.

Proposition 2.3.1: The idempotent element p in the ring Z_{2p} is center of the regular divisor graph $\mathfrak{R}_\partial(Z_{2p})$.

Proof: Since p is idempotent element in the ring, then $p^2 = p$. We have to prove that p has the maximum degree in the graph, p is regular element and its adjacent as a vertex with all vertices in the first partite set V_1 (odd or unit elements) of the vertex set $V(Z_{2p})$of the graph as follow:

for any $a_i \in V_1$, $p.a_i.p = p^2.a_i = p.a_i = p, a_i \neq p$

since a_i is odd, then $a_i = 2m+1$ for $m = 0,1,2,\dots$.

Then $p^2.a_i = p.a_i = p(2m+1) = 2mp + p = p$

since $V_1(Z_{2p})$ contains exactly p vertices, we exclude p, then the $deg(p) = p-1$. And p have no other adjacency.

Now we must calculate the degree of all other vertices in the graph $\mathfrak{R}_\partial(Z_{2p})$ as follow:

This ring Z_{2p} has another idempotent element $(p+1)$ and it is adjacent only with the vertex 1 in the regular divisor graph $\mathfrak{R}_\partial(Z_{2p})$, then deg(p+1)=1.

The vertices ai in V_1 and $b_i\ in\ V_2$ are adjacent together as follows.

In V_1 and V_2 , the unit elements are adjacent together each ai with a_i^{-1} and b_i with b_i^{-1} , but two elements 1 and $2p-1$ in V_1 are their own inverse and they are adjacent with p+1 and p-1 in V_2 respectively rather than the adjacency with vertex p as follow:

$(p+1).1.(p+1) = (p+1)^2.1 = (p+1)^2 = p+1$ (idempotent)

and $(p-1).(2p-1).(p-1) = (p-1)^2.(2p-1)$

$= (p^2-2p+1).(2p-1) = (2p^3-4p^2+2p-p^2+2p-1)$

$= 2p-4p+2p-p+2p-1$ $since\ p\ is\ idempotent\ p^2 = p$

$= 4p-4p-p+2p-1$ = (p-1). Then $deg(1) = deg(2p-1) = 2$.

In another hand two vertices p-1 and p+1 in V_2 are two end vertices, they are of degree one.

We proved that the vertex p has a maximum degree in the regular divisor graph, then p is center of the graph $\mathfrak{R}_\partial(Z_{2p})$.

Remark: To describe the regular divisor graph $\mathfrak{R}_\partial(Z_n)$ we need to define a new operation in graph theory we called it half join and denoted by $\oplus_r$ means the join operation between two graphs when we take half number of vertices in the second graph.

Definition 2.3.3 The half-join operation between graph $G(p_1,q_1)$ with r-regular graph $H(p_2,q_2)$ is defined by joining the vertices of the graph $G(p_1,q_1)$ with r numbers of vertices of r-regular graph $H(p_2,q_2)$ is denoted by $G \oplus_r H$ such that

$V(G \oplus_r H) = V(G) + V(H) = p_1 + p_2$

$E(G \oplus_r H) = E(G) + E(H) + \{uv_i : u \in G\ and\ v_i \in H, i = 1,2,\dots,r\} = q_1 + q_2 + \frac{p_1 p_2}{r}$

Theorem2.3.4: The regular divisor graph of the ring Z_{2p} (for all prime number $p \geq 3$) is

$\mathfrak{R}_\partial(Z_{2p}) \cong k_1(+_2 (2p_2 \cup \left(\frac{p-3}{2}\right) c_4)$

Proof: The vertex set of regular divisor graph $\mathfrak{R}_\partial(Z_{2p})$ except center p is partition

in to two partite sets relation with regularity property.
$V_1 - \{p\}$ contains all odd elements (unit elements), $V_1 = \{1,3,5,\dots\dots,2p-1\} - \{p\}$
V_2 contains all even elements, $V_2 = \{2,4,6,\dots,2(p-1)\}$, each sets contain exactly p-1vertices.
the elements in $V_1 - \{p\}$ are unit elements, then they are adjacent each together $a_i \ with \ a_i^{-1}$,$\forall \ a_i \in V_1 - \{p\}$. In the other hand $b_i \in V_2$ is adjacent with $a_i \in V_1 - \{p\}$ and is adjacent with $b_i^{\sim} \in V_2$ since $b_i = b_i.a_i.b_i$ and $b_i = b_i.b_i^{\sim}.b_i$.
$b_i^{\sim} \in V_2$ is adjacent with $a_i^{-1} \in V_1 - \{p\}$ and is adjacent with $b_i \in V_2$ since $b_i^{\sim} = b_i^{\sim}.a_i^{-1}.b_i^{\sim}$ and $b_i^{\sim} = b_i^{\sim}.b_i.b_i^{\sim}$, for all $i = 1,2,3,\dots,p-3$, these relations make the cycle C_4 (exactly $(\frac{p-3}{2})C_4$), but the two elements 1 and $2p-1$ in V_1 are adjacent with two elements $p+1$ and $p-1$ in V_2 respectability to makes the path P_2 since $(p+1)^2.1 = (p+1)^2 = p^2+2p+1 = p+1 \ since \ p \ is \ idempotent \ (p^2 = p)$
And $(p-1)^2.(2p-1) = (p-1)$.
Then now we have the part $(2p_2 \cup \left(\frac{p-3}{2}\right) c_4)$.
Now, the center is K_1 and since $p \in V_1$ (the center of graph) (proposition 5.1) and have maximum degree since it is adjacent with all other vertices in V_1 ($p = p.a.p$, $\forall \ a \in V_1$) then p is adjacent with one vertex of each path part P2 (we have 2p2) we get $p(+_2 \ 2p_2$, in the other hand p adjacent with 2 vertices of each cycle C4 (we have exactly $\frac{p-3}{2}$ c_4), so we get $k_1(+_2(2p_2 \cup \left(\frac{p-3}{2}\right) c_4)$ by the new operation, implies that $\mathfrak{R}_{\partial}(Z_{2p}) \cong k_1(+_2(2p_2 \cup \left(\frac{p-3}{2}\right) c_4)$ As shown in figure-2.4-

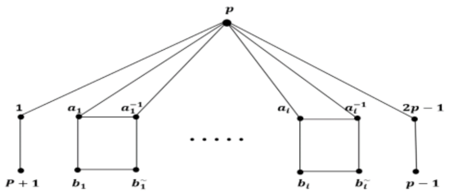

Figure 2.4: general form of the regular divisor graph R_∂ (Z_2p)

Remark: In figure-2.4- $b_1 = a_1^{-1} + p$, $b_1^{\sim} = a_1 + p$ and $b_i = a_i^{-1} + p$, $b_i^{\sim} = a_i + p$

Corollary 2.3.5: The regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$ has two end vertices for all p in the ring (Z_{2p}) .

Proof: The two vertices are $p - 1$ and $p+1$ are in the second partite set V_2 of vertex set $V(Z_{2p})$,

Since $p+1$ is one of the idempotent elements in the ring, then $(p+1)^2 = p+1$
up to the regularity of the $(p+1)$ its adjacent with only one vertex 1, and the other $p-1$ is regular with respect to $(2p-1)$ also we exclude the self-regularity by the same reason.

$(p-1)^2.(2p-1) = 2p^3 - 3p^2 + 4p - 1 = p - 1 \, , since \ (p$ is idempotent)

There exists an edge e_2 joins these two vertices and no other edges, $deg(p-1) = deg(p+1) = 1$

Proposition 2.3.6: The regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$ contains $(\frac{p-3}{2})$ cycles of order 4.

Proof: In the fact that we have always two end vertices $p+1$ and $p-1$ adjacent with
1 and $2p-1$ respectability and they are the only two vertices of degree one.

All the other vertices in $V_1 - \{p\}$ and V_2 are adjacent together as follow to make the cycle c_4

(a_i, a_i^{-1}) , $(b_i, b_i^{\sim})$, (a_i, b_i),and $(a_i^{-1}, b_i^{\sim})$

Since we have $p-1$ vertices in each partite set we exclude two vertices in each partite sets then we have $(\frac{p-3}{2})$ cycles of length 4.

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Corollary 2.3.7: The regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$ for p>2, is planner graph.

Proof: By theorem 2.3.4 clearly has no crossing number in the regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$ then it is planner graph.

Proposition 2.3.8: The clique number $\omega(G)$ of the regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$ is equal 3.

Proof: The regular divisor graph of the ring Z_{2p} is planner graph and the smallest cycle in $\mathfrak{R}_{\partial}(Z_{2p})$ is C_3 obtained from the adjacency between the vertices p, a_i, a_i^{-1} then complete subgraph is k_3 . And the order of k_3 is equal to 3.

Corollary 2.3.9: The clique number equal to girth in the graph $\mathfrak{R}_{\partial}(Z_{2p})$.

$$\omega\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = gi\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 3$$

Proof: It is clear that the shortest cycle in the graph $\mathfrak{R}_{\partial}(Z_{2p})$ is C_3 and length of this cycle is three then girth of the graph is equal to 3 and clique number=3.

Proposition 2.3.10: The dimeter of regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$,

$$Dim\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 4 .$$

Proof: In the general form of the graph $\mathfrak{R}_{\partial}(Z_{2p})$ that it is shown in the figure-2.4- it is clear that the distance between p with a_i for all $a_i \in V_1 - \{p\}$ is equal to 1, the distance between p with b_j for all $b_j \in V_2$ is equal to 2, the distance between a_i for all $a_i \in V_1 - \{p\}$ is equal to 1 or 2, the distance between a_i with b_j is equal to 1 or 2 or 3 for all $a_i \in V_1 - \{p\}$ and $b_j \in V_2$, the distance between b_j is equal to 1 or 4 for all $b_j \in V_2$, So, the maximum distance in the graph $\mathfrak{R}_{\partial}(Z_{2p})$ is equal to 4 , Then

$$Dim\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 4$$

Theorem 2.3.11: Chromatic number $\chi\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 3$.

Proof: The vertex p is adjacent with all vertices in $V_1 - \{p\}$ and for all $a_i \in V_1 - \{p\}$ there exists $a_i^{-1} \in V_1 - \{p\}$ such that a_i is adjacent with a_i^{-1}, then they must have different color, the vertices in V_2 they are adjacent together and adjacent with some vertices in V_1 by respect to the regularity $p+1, p-1$ are adjacent with $1, 2p-1$ respectability, then if p and $p+1, p-1$ are red all $a_i, b_j^{\sim}$ with 1 and $2p-1$ take another color say blue and a_i^{-1} with b_j take another color, so we use three different colors to coloring all vertices in the graph $\mathfrak{R}_{\partial}(Z_{2p})$ As shown in figure-2.5-. Then

$$\chi\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 3 .$$

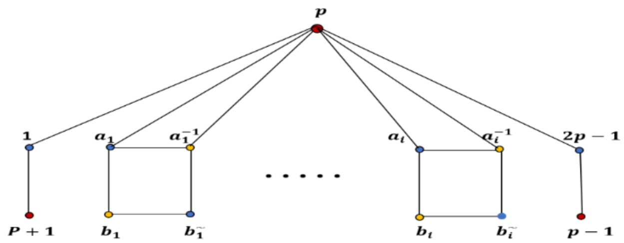

Figure 2.5: chromatic number for the general form in the regular divisor graph $\mathfrak{R}_{\partial}(Z_{2p})$.

Definition 2.3.12: Butterfly graph B_{1,nC_3} is a graph obtained from 2 path P_2 and n cycle C_3 identifying in one vertex r called a root as shown in figure-2.6-.

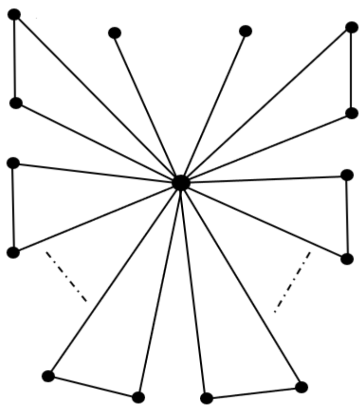

Figure 2.6: Butterfly graph B_{1,nC_3}

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Theorem 2.3.13: The regular divisor graph $\mathfrak{R}_\partial(Z_{2p})$ is Butterfly graph $B_{1,(\frac{p-3}{2})C_3}$ of order p and size $\frac{3p-5}{2}$ by removing all even vertices from the vertex set $V(Z_{2p})$.

Proof: Since the vertex set of regular divisor graph $\mathfrak{R}_\partial(Z_{2p})$ is partition in to two partite sets relation with regularity property.
$V_1 = \{1,3,5,\ldots\ldots,2p-1\}$
$V_2 = \{2,4,6,\ldots,2(p-1)\}$. $V_1 - \{p\} and\ V_2$ has exactly $p-1$ vertices. If we remove the even vertices and all the incident edges from the vertex set $V(Z_{2p})$ so only the first part of vertex set remains and since $p \in V_1$ is the center of graph (proposition 2.3.1) and have maximum degree since it is adjacent with all other vertices in V_1 ($p = p.a_i.p$, $\forall\, a_i \in V_1$) and the elements in $V_1 - \{p\}$ are unit elements, then they are adjacent each together $a_i\ with\ a_i^{-1}$,$\forall\, a_i \in V_1 - \{p\}$ then p with a_i and a_i^{-1} makes the cycle C_3 but the two elements 1 and 2p-1 in V_1 are self − regular then they do not adjacent with it is inverse and they are makes two path P_2 . Since $|V_1 - \{p\}| = p-1$ then $i = 1,2,3,\ldots,\frac{p-3}{2}$
Then the graph we got from V_1 is Butterfly graph $B_{1,(\frac{p-3}{2})C_3}$. As shown in the figure-2.7-

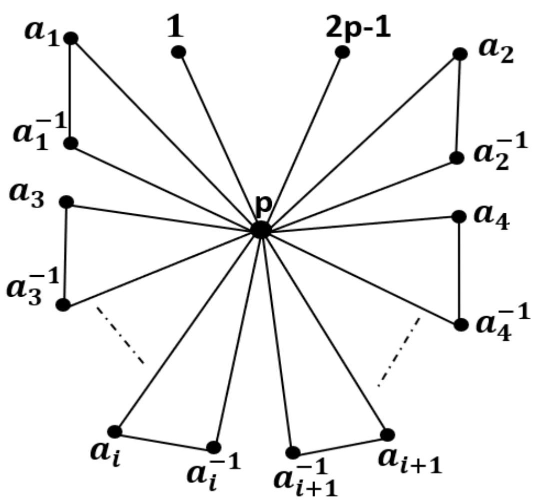

Figure 2.7

2.4 Regular divisor graph of the ring Z_{3p} for all prime number $p > 3$

The Regular divisor graph of the ring Z_{3p}, p is prime number and $p > 3$, different graph and has different properties.
In this section we will study the properties of the regular divisor graph $\mathfrak{R}_\partial(Z_{3p})$.
The ring Z_{3p} is regular ring for all prime number $p > 3$ and the regular divisor graph of this type of ring has different shape or special case since the vertex set $V(Z_{3p})$ of this ring is different from the vertex set of the ring Z_{2p} certainly we get a different graph with special cases. First, we give an example to show the regular divisor graph of Z_{3p} .

Example 6: Consider the ring Z_{15} , $q = 3\ and\ p = 5$
The vertex set $V(Z_{15}) = \{1,2,3,\ldots,14\}$)
the regular divisor graph $\mathfrak{R}_\partial(R)$ of the ring $R = Z_{15}$ is shown in figure -2.8-

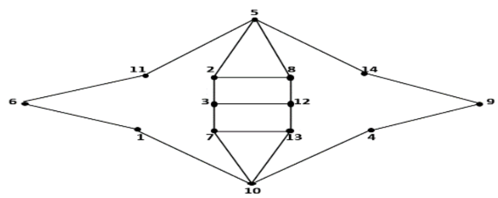

Figure 2.8: Regular divisor graph $\mathfrak{R}_\partial(z_{15})$

The properties of the regular divisor graph $\mathfrak{R}_\partial(z_{15})$

1) Center $\mathfrak{R}_\partial(z_{15}) = \{p, 2p\}$ since $deg(p) = deg(2p) = p-1$
2) $P + 1,2p = (6,10)$ are idempotent elements
3) The vertices $4,5,6,9,10,11 = \{p-1, p, p+1, 2p-1, 2p, 2p+1\}$ are two sides regular.
4) $gi(\ \mathfrak{R}_\partial(z_{15})\ ) = \omega(\mathfrak{R}_\partial(z_{15})\ ) = 3$
5) The regular divisor graph $\mathfrak{R}_\partial(z_{15})$ is planner connected graph
6) $Dim\ (\mathfrak{R}_\partial(z_{15})\ ) = 4$
7) This graph contains one circuit ($c_4 \bullet\bullet c_4$) of order 6
8) $|\mathfrak{R}_\partial(z_{15})\,| = 14$, $E(\mathfrak{R}_\partial(z_{15})) = 19$

To explain how we study the cases of the regular divisor graph of the ring Z_{3p} , we have to give another example to show the different graphs with respect to the adjacency between vertices.

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Example 7 The regular divisor graph of the ring $Z_{3.7} = Z_{21}$

$V(Z_{21}) = \{1,2,3,4, \dots ,20\}$

$and \ E\left(\mathfrak{R}_{\partial}(z_{15})\right) = \{(a,b)\ , a = a.b.a \ or \ b = b.a.b \ \ \forall \ a \neq b \neq 0 \in Z_{21}\}$

The regular divisor graph of the ring Z_{21} is shown in figure -2.9-

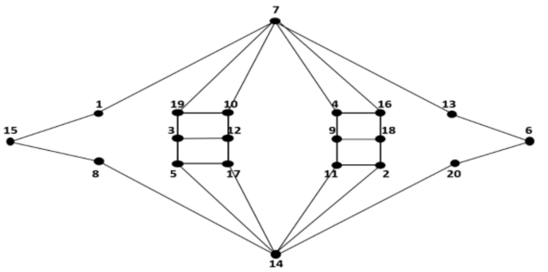

Figure 2.9: Regular divisor graph $\mathfrak{R}_{\partial}(z_{21})$

The properties of the regular divisor graph $\mathfrak{R}_{\partial}(z_{21})$

1) Center = $\{\ p, 2p\} = \{7,14\}$ sine $deg(p) = deg(2p) = 6$
2) $P \ and \ 2p + 1$ are idempotent
3) Self-regular elements $\{6,7,8,13,14,15\} = \{p - 1, p, p + 1,2p - 1,2p, 2p + 1\}$ are loops
4) $gi(\mathfrak{R}_{\partial}(z_{21}))= \omega(\mathfrak{R}_{\partial}(z_{21}))=3$
5) $Dim\ (\mathfrak{R}_{\partial}(z_{21}))$ =4
6) The regular divisor graph $\mathfrak{R}_{\partial}(z_{21})$ is connected planner graph
7) This graph contains two circuit $(c_4 \bullet\bullet c_4)$ of order 6

When we compare these two rings in the examples 6 and 7 and study the properties of their regular divisor graphs in figure -2.11- and figure -2.12- we get the following:

1) In each case the graph has two centers of greatest degree they are $p \ and \ 2p$
2) The graph in each case connected planner, grith and clique =3 , with diameter =4.
3) Order of graph $\left|\mathfrak{R}_{\partial}(z_{3p})\right| = 3p - 1 \ \ and \ \ E\left(\mathfrak{R}_{\partial}(z_{15})\right) = \frac{(11p-17)}{2}$
4) The idempotent elements are different in each case, when

$e = \{p + 1,2p\}$ in Z_{15} for $q = 3 \ and \ p = 5$.

and $e^{\sim} = \{p, 2p + 1\}$ in Z_{21} for $q = 3 \ and \ p = 7$

in $\mathfrak{R}_{\partial}(z_{15})$ the first idempotent element $p + 1 = 6$ is adjacent with $1 \ and \ 11 = 2p + 1$

While the second idempotent $2p = 10$ is adjacent with $1 \ and \ p - 1 = 4$

But in the ring Z_{21} , the idempotent element 2p+1=15 is adjacent with vertices 1 and p+1=8 ,

And the second p=7 is adjacent with 1 and 2p-1=13.

For this reason and depending on the adjacency between the vertices as regular element in the ring Z_{3p} in general we have two cases.

Now to study the regular divisor graph of the ring Z_{3p}, we should make this study in to two different cases according to the adjacency of idempotent elements as a vertex in this graph.

For these two cases we need to give the following figures:

figure-2.10- and figure-2.11-, shows two general cases of regular divisor graph of the ring z_{3p}

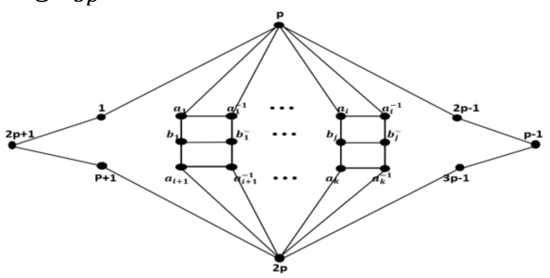

Figure 2.10: General form of Regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$

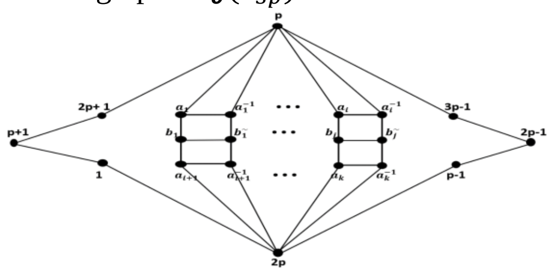

Figure 2.11: General form of Regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$

As shown in the figure 2.10 and figure 2.11 In general, the vertex set V of the regular divisor graph of the ring Z_{3p} is:

$V(\mathfrak{R}_{\partial}(z_{3p}))= \{1, 2, 3, \dots \dots , 3p - 1\}$

Doi: https://doi.org/10.25130/tjps.v28i5.1587

To justify this, we partition this set of regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$ into three partite sets as follow:

$V_1(Z_{3p}) = \{1, p, 2p, p-1, 2p-1, 3p-1, p+1, 2p+1\}$ these vertices make border of the graph, then they are vertices of circumference of graph, some elements in V_1 are self-unit and others are non-unit.

Two vertices p and 2p in V_1 are centers and have maximum degree $(p-1)$ and the other vertices of degree 2.

The second partite set $V_2(Z_{3p})$ = $\{a_i: a_i$ is unit element for all i=1, 2,…. 2(P-3)} and they are exactly $(2p-6)$ vertices in the regular divisor graph.

$V_3(Z_{3p}) = \{\ b_j\ ,\ j = 1, 2, \ldots (p-3)$, all other non-unit elements such that they are adjacent together by regularity} ={ $3k, k = 1, 2, \ldots (p-1)$}

Then $V(Z_{3p}) = V_1(Z_{3p}) \cup V_2(Z_{3p}) \cup V_3(Z_{3p})$

$|V(Z_{3p})| = 3p - 1$

$|V_1(Z_{3p})| = 8$

$|V_2(Z_{3p})| = 2(p-3)$

$|V_3(Z_{3p})| = p - 3$

Now we discuss the two cases for regular divisor graph $\mathfrak{R}_{\partial}(z_{qp})$ for all prime number $q = 3$ and $p > 3$ as follow:

Case1: For q=3 and p=7,13,19, …

In this case p as a center is adjacent with the vertices $\{1,4,7, \ldots, 3p-2\} - p \subset V_2$ except $\{1, 2p-1\} \subset V_1$ and $2p$ is adjacent with the vertices $\{2,5,8, \ldots, 3p-1\} - 2p \subset V_2$ except $\{p+1, 3p-1\} \subset V_1$ and all elements in V_2 are unit they are adjacent each element with its inverse. In this case p and $2p+1$ are idempotents $[p^2 = p\ and\ (2p+1)^2 = 2p+1]$ but $2p$ is not idempotent and $(2p)^2 = p$, but the vertices $1, 2p-1, 3p-1, p+1$ in V_1 are self-inverse and the vertices $p-1, 2p+1$ in V_1 are non-unit elements.

Proposition 2.4.1: In case1: $deg(2p+1) = deg(p-1) = 2$ such that:

i) The vertex $2p+1 \in V_1$ is adjacent with the vertices $1\ and\ p+1\ in\ V_1$

ii) The vertex $p-1 \in V_1$ is adjacent with the vertices $2p-1\ and\ 3p-1\ in\ V_1$

Proof:

i) According to the regularity $(2p+1)^2.1 = (2p+1).1$ $=2p+1$

And $(2p+1)^2.(p+1) =$ $= (2p+1).(p+1)$

$= 2p^2 + 2p + p + 1$ (p is idempotent)

$= 3p + 2p + 1$

$=2p+1$

Then the vertex $(2p+1)$ is adjacent with two vertices 1 and (p+1)

ii)In the other hand (p-1) is regular with respect to two elements (2p-1) and (3p-1), then

$(p-1)^2.(2p-1) = (p^2 - 2p + 1).(2p-1)$

$=(p - 2p + 1).(2p-1)$ since in this case p is idempotent $(p^2 = p)$

$= (1-p).(2p-1)$ $= 2p - 1 - 2p^2 + p$ $=p-1$

And $(p-1)^2.(3p-1)$ $= (p^2 - 2p + 1).(3p-1)$ (p is idempotent $p^2 = p$)

$=(1-p).(3p-1)$ $=3p - 1 - 3p^2 + p$

$=p-1$

Then we get the adjacency of this vertex with two vertices (2p-1) and (3p-1) to get the result.

Case2: For $q = 3\ and\ p = 5, 11, 17, \ldots$. n this case $2p$ and $p+1$ are idempotents $[(2p)^2 = 2p\ and\ (p+1)^2 = p+1]$

But p is not idempotent and $p^2 = 2p$. in this case p is adjacent with the vertices $\{2,5,8, \ldots, 3p-1\} - p \subset V_2$ except $\{2p+1, 3p-1\} \subset V_1$ and $2p$ is adjacent with the vertices $\{1,4,7, \ldots, 3p-2\} - 2p \subset V_2$ except $\{1, p-1\} \subset V_1$ and all elements in V_2 are adjacent with its inverse.

In this case the vertices $1, p-1, 3p-1, 2p+1$ in V_1 are self-inverse then there are no edges joins them.

Proposition 2.4.2: In case2

i)The vertex $p+1 \in V_1$ is adjacent with the vertices $1\ and\ 2p+1\ in\ V_1$

ii)The vertex $2p-1 \in V_1$ is adjacent with the vertices $p-1\ and\ 3p-1\ in\ V_1$,

Then $deg(p+1) = deg(2p-1) = 2$

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Proof:

i) $(p+1)^2.1=(p+1)$ since in this case $p+1$ is idempotent

And $(p+1)^2.(2p+1)=(p+1).(2p+1)$ (also $p+1$ is idempotent)

$=2p^2+p+2p+1 = 2(2p)+p+2p+1$ $since\ p^2=2p$

$=4p+p+2p+1 = 6p+p+1 = p+1$

ii) $(2p-1)^2.(p-1)=((2p)^2-4p+1).(p-1)$

$=(2p-4p+1).(p-1)$ $since\ in\ this\ case\ (2p)^2=2p$

$=2p^2-4p^2+p-2p+4p-1$

$=2(2p)-2p+p-2p+4p-1$ since $4p^2=(2p)^2=2p\ and\ p^2=2p$

$=2p-1$

And $(2p-1)^2.(3p-1)= ((2p)^2-4p+1).(3p-1)$

$=(2p-4p+1).(3p-1)$ $since\ in\ this\ case\ (2p)^2=2p$

$=6p^2-12p^2+3p-2p+4p-1$ $=6(2p)-12(2p)+3p-2p+4p-1$ $(p^2=2p)$

$=12p-24p+5p-1=5p-1=2p-1$

It is worth mentioning in both cases for all $b \in V_3$ there is two elements in V_2 such that b is adjacent with them.

Proposition 2.4.3: The regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$ contains $\frac{(p-3)}{2}$ subgraphs of the for circuit $(C_4{\bullet\bullet}C_4).(C_4{\bullet\bullet}C_4$ denoted the identifying an edge between two cycles)

Proof: As it appears in figures 2.10 and 2.11 the vertices in V_2 be divided into two parts the vertices of one of the parts are adjacent with the vertex p and adjacent with a vertex in V_3 such that a_i with b_j and a_i^{-1} with $b_j^{\sim}$ the vertices of the other part in V_2 are adjacent with the vertex 2p and adjacent with a vertex in V_3 such that a_k with b_j and a_k^{-1} with $b_j^{\sim}$, in both parts they are adjacent together a_i with a_i^{-1} , in the other hand the vertices in V_3they are adjacent together by regularity b_j with $b_j^{\sim}$then one of the part in V_2 makes a cycles C_4 as follow

$(a_i,a_i^{-1}),(a_i,b_j),(a_i^{-1},b_j^{\sim}),(b_j,b_j^{\sim})$ for some $a_i,a_i^{-1} \in V_2$ and $b_j,b_j^{\sim} \in V_3$

And another part of V_2 with the vertices in V_3 makes the cycles C_4 as follow

$(a_k,a_k^{-1}),(a_k,b_j),(a_k^{-1},b_j^{\sim}),(b_j,b_j^{\sim})$ for some $a_k,a_k^{-1} \in V_2$ and $b_j,b_j^{\sim} \in V_3$

The edge $(b_j,b_j^{\sim})$ is identifying between both cycles then we get the circuit $C_4{\bullet\bullet}C_4$ since $|V_2(z_{3p})|=2p-6$ and $|V_3(z_{3p})|=p-3$ so we have exactly $\frac{(p-3)}{2}$ circuit $C_4{\bullet\bullet}C_4$.

Corollary 2.4.4 The regular divisor graph of the commutative ring Z_{3p} is connected planner graph.

Proof: It is clear in figure-2.10- and figure-2.11-has no crossing number in the graph $\mathfrak{R}_{\partial}(z_{3p})$ and all vertices are adjacent, then this graph is connected and planner graph.

Definition 2.4.5: Let G and H be two graphs the **inserting edge** between two graphs is denoted by $G : H$ if $e=uv$ is an edge joins a vertex $v \in G$ with a vertex $u \in H$ such that

$V(G{:}H)= V(G)+V(H)$

$E(G : H)$=$E(G)+E(H)+1$, and ⁝ denoted the inserting of two edges between them such that

$V(G ⁝ H)$= $V(G)+V(H)$, $E(G ⁝ H)$=$E(G)+E(H)+2$

Theorem 2.4.6: The regular devisor graph of the ring Z_{3p} is a new graph of the form $\mathfrak{R}_{\partial}(Z_{3p}) \cong$ p ⁝ H_i ⁝ 2p, where p and 2p are belong to the boarder part C_8, and H_i, i=1, 2, ..., p- 3/2 are isomorphic subgraphs of the form $(C_4{\bullet\bullet}C_4)$.(⁝ is denoted an inserting of two edges from p or 2p to Hi).

Proof: (Case 1)

It is clear that in case one the vertex p is adjacent with $1,2p-1$ and the vertex 2p is adjacent with $p+1,p-1,$ by proposition 2.4.1 the vertex $2p+1$ is adjacent with the vertices $1,p+1$ and the vertex $p-1$ is adjacent with the vertices $2p-1,3p-1$ then the vertices $\{p,2p,1,2p-1,p+1,3p-1,2p+1,p-1\}$ make a cycle C_8 (border of the graph) and by proposition 2.4.3 we have exactly $\frac{(p-3)}{2}$ subgraphs $(C_4{\bullet\bullet}C_4)$ since the vertex p is adjacent with $\{1,4,7,\dots,3p-2\}-p \subset$

V_2 and $2p$ is adjacent with the vertices $\{2,5,8,\dots,3p-1\}-2p \subset V_2$ then p and 2p inserting two edges to the vertices a_i, a_i^{-1} in V_2 and a_i, a_i^{-1} is a part of H_i then $\mathfrak{R}_{\partial}(Z_{3p)} \cong p : H_i : 2p$

For (case2):

It is clear that in case two the vertex p is adjacent with $2p+1,3p-1$ and the vertex 2p is adjacent with $1, p-1$, by proposition 2.4.2 the vertex $p+1$ is adjacent with the vertices $1,2p+1$ and the vertex $2p-1$ is adjacent with the vertices $p-1,3p-1$ then the vertices $\{ p, 2p, 1,2p-1, p+1,3p-1,2p+1, p-1\}$ make a cycle C_8 (border of the graph) and by proposition 2.4.3 we have exactly $\frac{(p-3)}{2}$ subgraphs $(C_4 \bullet\bullet C_4)$ since the vertex p is adjacent with $\{2,5,8,\dots,3p-1\}-p \subset V_2$ and $2p$ is adjacent with the vertices $\{1,4,7,\dots,3p-2\}-2p \subset V_2$ then p and 2p inserting two edges to the vertices a_i, a_i^{-1} in V_2 and a_i, a_i^{-1} is a part of H_i then $\mathfrak{R}_{\partial}(Z_{3p)} \cong p : H_i : 2p$.

Proposition 2.4.7: The regular divisor graph of the ringZ_{3p}, $\mathfrak{R}_{\partial}(Z_{3p})$ is double butterfly graph by removing the non-unit vertices except $\{p, 2p\}$from the vertex set of the graph.

$$\mathfrak{R}_{\partial}(Z_{3p)} \cong 2B_{1,nC_3}\ ,\ n=\frac{p-3}{2}$$

i- for V($\mathfrak{R}_{\partial}(Z_{3p)}$)- $V_3 \cup \{p-1,2p+1\}$ in case one.

ii- or V($\mathfrak{R}_{\partial}(Z_{3p)}$)- $V_3 \cup \{p+1,2p-1\}$ in case two.

Proof: The vertex set of the graph ($\mathfrak{R}_{\partial}(Z_{3p)}$ is three partite sets as follow:

$V_1(Z_{3p}) = \{1, p, 2p, p-1,2p-1,3p-1, p+1,2p+1\}$

$V_2(Z_{3p}) = \{a_i : a_i$ is unit element for all $i = 1,2, \dots . 2(P-3)\}$

$V_3(Z_{3p}) = \{\ b_j\ ,\ j = 1, 2, \dots\ (p-3)$, all other non-unit elements$\}$ $=\{\ 3k, k = 1, 2, \dots (p-1)\}$

In case one

The nun unit vertices except {p,2p} are equal to the vertices in $V_3 \cup \{p-1,2p+1\}$ and by removing these vertices remain the vertices $\{p, 2p, 1,2p-1, p+1,3p-1\} \subset V_1$ with all vertices in V_2 , since in case one the vertex p is adjacent with the vertices $\{1,4,7,\dots,3p-2\}-p \subset V_2$ except $\{1,2p-1\} \subset V_1$and $2p$ is adjacent with the vertices $\{2,5,8,\dots,3p-1\}-2p \subset V_2$ except $\{p+1,3p-1\} \subset V_1$, then p with the vertices $1,2p-1$ make two paths P_2 and 2p with the vertices $p+1,3p-1$ make two paths P_2 and all other vertices that are adjacent with p and 2p they are in V_2 and they are unit each $a_i \in V_2$ is adjacent with $a_i^{-1} \in V_2$, So, (a_i, p, a_i^{-1}) and $(a_i, 2p, a_i^{-1})$ make the cycles C_3 then we get two butterfly by removing all nun-unit elements except $p, 2p$ in the graph ($\mathfrak{R}_{\partial}(Z_{3p)}$), as shown in the figure-2.12-

In case two

The nun unit vertices except {p,2p} are equal to the vertices in $V_3 \cup \{p+1,2p-1\}$ and by removing these vertices remain the vertices $\{p, 2p, 1, p-1,2p+1,3p-1\} \subset V_1$ with all vertices in V_2 , since in case two the vertex p is adjacent with the vertices $\{2,5,8,\dots,3p-1\}-p \subset V_2$ except $\{2p+1,3p-1\} \subset V_1$ and $2p$ is adjacent with the vertices $\{1,4,7,\dots,3p-2\}-2p \subset V_2$ except $\{1, p-1\} \subset V_1$, then p with the vertices $2p+1,3p-1$ make two paths P_2 and 2p with the vertices $1, p-1$ make two paths P_2 and all other vertices that are adjacent with p and 2p they are in V_2 and they are unit each $a_i \in V_2$ is adjacent with $a_i^{-1} \in V_2$, So, (a_i, p, a_i^{-1}) and $(a_i, 2p, a_i^{-1})$ make the cycles C_3 then we get two butterfly by removing all nun-unit elements except $p, 2p$ in the graph ($\mathfrak{R}_{\partial}(Z_{3p)}$), as shown in the figure-2.13-

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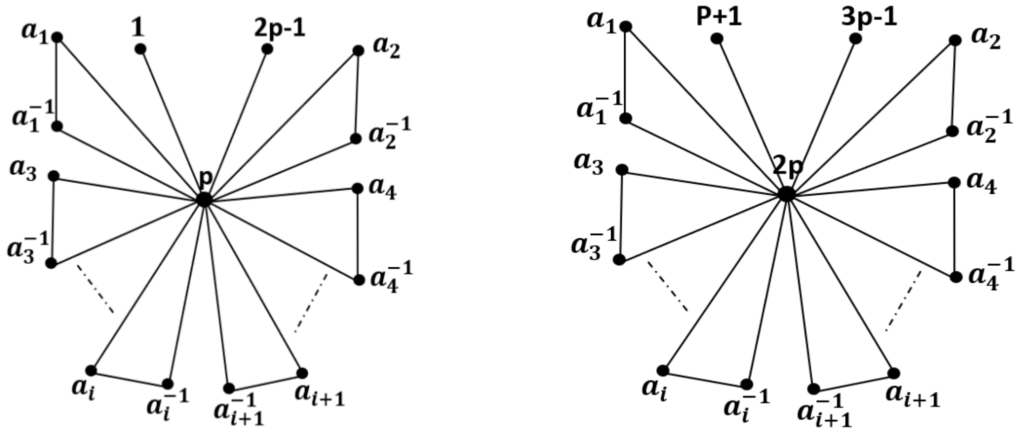

Figure 2.12

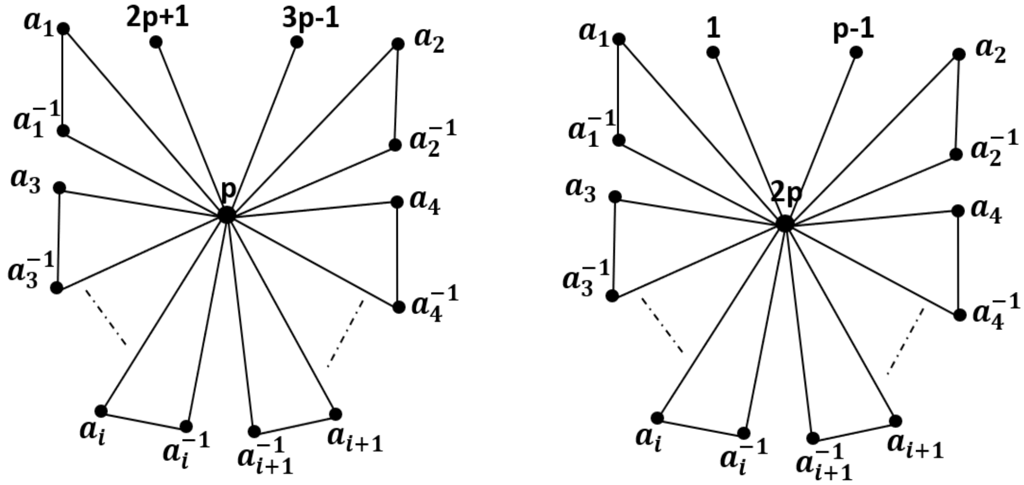

Figure 2.13

Corollary 2.4.8: The clique number of regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$ is equal to 3.

$\omega(\mathfrak{R}_{\partial}(z_{3p})) = 3$

Proof: From the fact that this graph is planner graph and the smallest cycle in regular divisor graph $\mathfrak{R}_{\partial}(z_{3p})$ is C_3 obtained from the adjacency between the vertices p, a_{i-1}, a_{i-1}^{-1} and $2p, a_i, a_i^{-1}$ for all $a_{i-1}, a_{i-1}^{-1}, a_i, a_i^{-1} \in V_2$ then the smallest complete subgraph is k_3 .

Theorem 2.4.9 Chromatic number $\mathcal{X}\left(\mathfrak{R}_{\partial}(Z_{3p})\right) = 3$

Proof: For case1 we give the same color to coloring the vertices $p, 2p, 2p+1, p-1, \{b_1, \dots, b_j\}$ and we use another color to coloring the vertices $1, 2p-1, p+1, 3p-1, \{a_1^{-1}, \dots, a_i^{-1}, a_{i+1}^{-1}, \dots, a_k^{-1}\}$, we use another color to coloring the vertices $,\{a_1, \dots, a_i, a_{i+1}, \dots a_k\}, \{b_1^{\sim}, \dots,, b_j^{\sim}\}$,

for case2 we give the same color to coloring the vertices $p, 2p, p+1, 2p-1, \{b_1, \dots, b_j\}$ and we use another color to coloring the vertices $1, 2p+1, 3p-1, p-1, \{a_1^{-1}, \dots, a_i^{-1}, a_{i+1}^{-1}, \dots, a_k^{-1}\}$, give another color to coloring the vertices $\{a_1, \dots, a_i, a_{i+1}, \dots a_k\}, \{b_1^{\sim}, \dots,, b_j^{\sim}\}$, so in both cases we use only three different colors to coloring all vertices in $\mathfrak{R}_{\partial}(Z_{3p})$. As shown in the figure-2.14-Then the chromatic number of $\mathfrak{R}_{\partial}(Z_{3p})$ is equal to 3.

$\mathcal{X}\left(\mathfrak{R}_{\partial}(Z_{3p})\right) = 3$

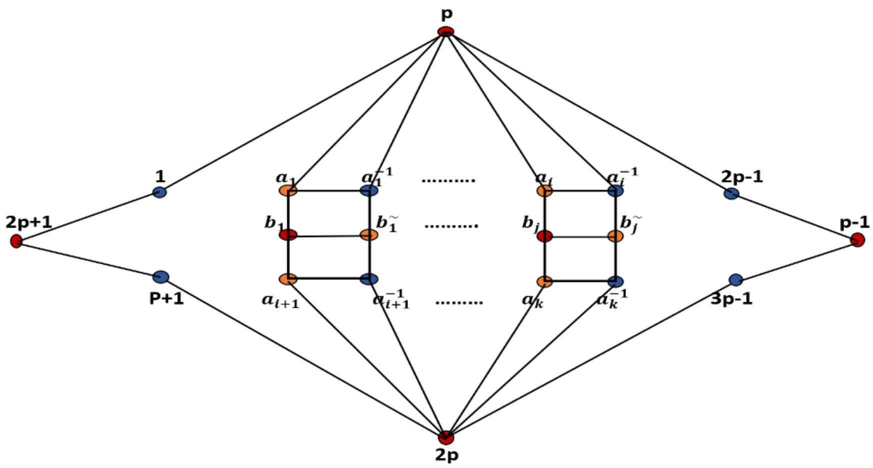

Figure 2.14: chromatic number for the general form in the regular divisor graph $\mathfrak{R}_\partial(Z_{3p})$.

Proposition 2.4.10: Dimeter of regular divisor graph $\mathfrak{R}_\partial(Z_{3p})$ $\mathcal{Dim}\left(\mathfrak{R}_\partial(Z_{3p})\right) = 4$

Proof: For case 1, in the general form of the graph $\mathfrak{R}_\partial(Z_{3p})$ that is shown in the figure-2.10- $V_1(Z_{3p}) = \{1, p, 2p, p-1, 2p-1, 3p-1, p+1, 2p+1\}$ the distance between the vertices in V_1 are
$d(p,2p) = 4$, $d(p,1) = d(p,2p-1) = 1$,
$d(p,p+1) = d(p,3p-1) = 3$, $d(p,2p+1) = 2$,
$d(p,p-1) = 2$, $d(2p,1) = 3$,
$d(2p,2p-1) = 3$,
$d(2p,p+1) = 1$, $d(2p,3p-1) = 1$,
$d(2p,2p+1) = 2$,
$d(2p,p-1) = 2$, $d(1,2p-1) = 2$,
$d(1,p+1) = 2$,
$d(1,3p-1) = 4$, $d(1,2p+1) = 1$,
$d(1,p-1) = 3$,
$d(2p-1,p+1) = 4$, $d(2p-1,3p-1) = 2$, $d(2p-1,2p+1) = 3$, $d(2p-1,p-1) = 1$, $d(p+1,3p-1) = 2$,
$d(p+1,2p+1) = 1$, $d(p+1,p-1) = 3$,
$d(3p-1,2p+1) = 3$, $d(3p-1,p-1) = 1$,
$d(2p+1,p-1) = 4$.
The distance between p with a_i equal to 1 or 3 for all $a_i \in V_2$, the distance between 2p with a_i equal to 1 or 3 for all $a_i \in V_2$, the distance between p with b_j equal to 2 for all $b_j \in V_3$, the distance between 2p with b_j equal to 2 for all $b_j \in V_3$,the distance between a_i equal to 1 or 2 or 3 or 4 for all $a_i \in V_2$, the distance between b_j equal to 1 or 4 for all $b_j \in V_3$, the distance between a_i with b_j equal to 1 or 2 or 3 for all $a_i \in V_2$ and $b_j \in V_3$,

And for case 2
in the general form of the graph $\mathfrak{R}_\partial(Z_{3p})$ that is shown in the figure-2.11- $V_1(Z_{3p}) = \{1, p, 2p, p-1, 2p-1, 3p-1, p+1, 2p+1\}$
the distance between of vertices in V_1 are
$d(p,2p) = 4$, $d(p,1) = 3$, $d(p,2p-1) = 2$,
$d(p,p+1) = 2$, $d(p,3p-1) = 1$, $d(p,2p+1) = 1$,
$d(p,p-1) = 3$, $d(2p,1) = 1$,
$d(2p,2p-1) = 2$,
$d(2p,p+1) = 2$, $d(2p,3p-1) = 3$,
$d(2p,2p+1) = 3$,
$d(2p,p-1) = 1$, $d(1,2p-1) = 3$,
$d(1,p+1) = 1$,
$d(1,3p-1) = 4$, $d(1,2p+1) = 2$,
$d(1,p-1) = 2$,
$d(2p-1,p+1) = 4$, $d(2p-1,3p-1) = 1$, $d(2p-1,2p+1) = 3$, $d(2p-1,p-1) = 1$, $d(p+1,3p-1) = 3$,
$d(p+1,2p+1) = 1$, $d(p+1,p-1) = 3$,
$d(3p-1,2p+1) = 2$, $d(3p-1,p-1) = 2$,
$d(2p+1,p-1) = 4$.
The distance between p with a_i equal to 1 or 3 for all $a_i \in V_2$, the distance between 2p with a_i equal to 1 or 3 for all $a_i \in V_2$, the

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distance between p with b_j equal to 2 for all $b_j \in V_3$, the distance between 2p with b_j equal to 2 for all $b_j \in V_3$,the distance between a_i equal to 1 or 2 or 3 or 4 for all $a_i \in V_2$, the distance between b_j equal to 1 or 4 for all $b_j \in V_3$, the distance between a_i with b_j equal to 1 or 2 or 3 for all $a_i \in V_2$ and $b_j \in V_3$

So, the maximum distance in the graph $\mathfrak{R}_{\partial}(Z_{3p})$ equal to 4, then $\mathcal{Dim}\left(\mathfrak{R}_{\partial}(Z_{3p})\right) = 4$

1. Connectivity of the regular divisor graph for finite commutative rings.

A graph G is **connected** if there exists at least one path between any pair of vertices in G other wise is called **disconnected graph.** As shown in figure-3.1-, If G is a disconnected graph **component of** G is a maximal connected subgraph of G, number of components in graph G is denoted by $C(G)$. $C(G)$ is one if G is connected.

For any connected graph, a vertex u from G is named a **cut-vertex** of G, if $G - u$ (remove u from G) outcomes a disconnected graph. A proper subset $\bar{V} \in V$ is a **vertex cut set** if the graph $G - \bar{V}$ is disconnected, or trivial graph. The **vertex connectivity** of a connected graph G is the smallest number of vertices whose removal makes G disconnected or trivial graph and denoted by $K(G)$, the graph is said to be **k-vertex connected** or **k-connected** when $K(G)$ is the smallest size of a cut set of G it means $|\bar{V}| = k$.

And an edge e from a connected graph G is named a **cut-edge(bridge)** of G if $G - e$ (remove e from G) outcomes a disconnected graph. A proper subset $\bar{E} \subset E$ is **edge cut-set** if the graph $G - \bar{E}$ is disconnected. The **edge connectivity** of connected graph G is the smallest number of edges whose removal makes G disconnected and denoted by $\lambda(G)$. G is said to be m-edge connected if $\lambda(G)$ is the smallest size of edge cut-set, it means $|\bar{E}| = m$ as shown in figure-3.2-

The subgraph H of the graph G is known a **Block** if H is connected maximal subgraph of G which has no cut-vertex and the graph G is called **Block itself** if which has no cut-vertices.[8],[11].

In this section we denote the minimum degree vertex of the graph $\mathfrak{R}_{\partial}(z_n)$ by $\delta(\mathfrak{R}_{\partial}(z_n))$, the vertex connectivity of $\mathfrak{R}_{\partial}(z_n)$ is denoted by $K(\mathfrak{R}_{\partial}(z_n))$, and $\lambda(\mathfrak{R}_{\partial}(z_n))$ is the edge connectivity of $\mathfrak{R}_{\partial}(z_n)$

Theorem 3.1: In the regular divisor graph for the ring (Z_{2p}) (p is prime number and $p \geq 3$) has only one cut-vertex, then is 1-connected graph.

Proof: Since p is center of the graph $\mathfrak{R}_{\partial}(Z_{2p})$ which is greatest degree $deg(p) = p - 1$, it is clear that by removing the vertex p in $\mathfrak{R}_{\partial}(Z_{2p})$ we get the following graph show in the figure-3.1-

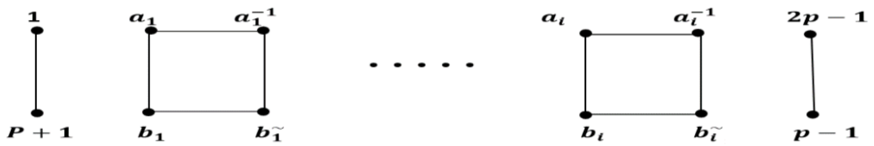

Figure 3.1: cut-vertex in $\mathfrak{R}_{\partial}(Z_{2p})$

And this graph is disconnected graph, then $\mathfrak{R}_{\partial}(Z_{2p})$ has only one cut-vertex and it is 1-connected graph.

Theorem 3.2: The graph $\mathfrak{R}_{\partial}(Z_{3p})$ (p is prime number and $p \geq 5$) is 2-connected graph.

Proof: Since p and 2p are centers of the graph $\mathfrak{R}_{\partial}(Z_{3p})$ which are greatest degree $deg(p) = p - 1$ and $deg\ (2p) = p - 1$, it is clear that by removing the vertices p and 2p in $\mathfrak{R}_{\partial}(Z_{3p})$(that shows in figure-2.10- and figure-2.11-) we get the following

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graph show in the figure-3.2- and figure-3.3-

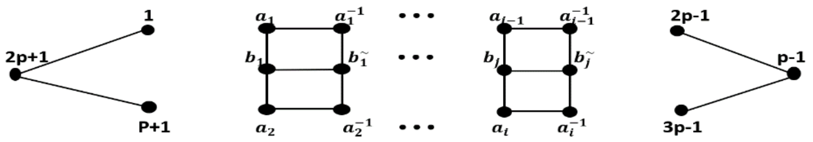

Figure 3.2

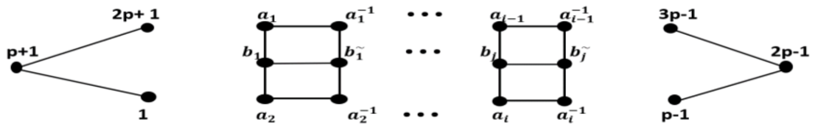

Figure 3.3

And these graphs are disconnected graphs, have two cute-vertices, then $\mathfrak{R}_{\partial}(Z_{3p})$ is 2-connected graph.

Theorem 3.3: The graph $\mathfrak{R}_{\partial}(Z_{2p})$ has only one cut-edge(bridge) and 1-edge connected.

Proof: The graph $\mathfrak{R}_{\partial}(Z_{2p})$ is connected graph, has two types of vertices set $V_1 = \{1,3,5,\dots,2p-1\}$ and $V_2 = \{2,4,6,\dots,2(p-1)\}$, and has four types of edges with respect to the regularity for the ring Z_{2p}.

Type one is the edges (p, a_i) for all $a_i \in V_1 - \{p\}$ since $p \in V_1$ is the center of graph (proposition 2.3.1) and have maximum degree since it is adjacent with all other vertices in V_1 $(p = p.a_i.p\ ,\ \forall\ a_i\ \in\ V_1 - \{p\}\ )$.

Type two is the edges (a_i, a_i^{-1}) for all $a_i, a_i^{-1} \in V_1 - \{p\}$ since the elements in $V_1 - \{p\}$ are unit elements, then they are adjacent each together $a_i\ with\ a_i^{-1}$,$\forall\ a_i \in V_1 - \{p\}$.

Type three is the edges (a_i, b_i) for all $a_i \in V_1 - \{p\}$ and $b_i \in V_2$ since $b_i \in V_2$ is adjacent with $a_i \in V_1 - \{p\}$, $b_i = b_i.a_i.b_i$.

Type four is the edges $(b_i, b_i^{\sim})$ for all b_i and $b_i^{\sim} \in V_2$ since $b_i \in V_2$ is adjacent with $b_i^{\sim} \in V_2$ since $b_i = b_i.b_i^{\sim}.b_i$.

If we remove any edge from type two or type three or type four the graph $\mathfrak{R}_{\partial}(Z_{2p})$ still connected but if we remove the edge $(p,1)$ or $(p,2p-1)$ on type one we get the new disconnected graph then the edge $(p,1)$ or $(p,2p-1)$ is bridge then the graph $\mathfrak{R}_{\partial}(Z_{2p})$ has only one cut-edge (bridge) and 1-edge connected.

Theorem 3.4: The graph $\mathfrak{R}_{\partial}(Z_{3p})$ is 2-edge connected.

Proof: If we look at figure-2.10- and figure-2.11- in which them the general form of the graph $\mathfrak{R}_{\partial}(Z_{3p})$ is shown. We see that by removing the edges $(p,1)$ and $(2p,p+1)$ or the edges $(p,2p-1)$ and $(2p,3p-1)$ in the first case and removing the edges $(p,2p+1)$ and $(2p,1)$ or the edges $(p,3p-1)$ and $(2p,p-1)$ in the second case we will get a new graph that is a disconnected graph, this is the minimum number of edges that removing in the graph $\mathfrak{R}_{\partial}(Z_{3p})$ we can get a disconnected graph. Then the graph $\mathfrak{R}_{\partial}(Z_{3p})$ has two cut-edges, then the graph $\mathfrak{R}_{\partial}(Z_{3p})$ is 2-edge connected.

Remark 3.5

The graph $\mathfrak{R}_{\partial}(Z_{2p})$ is connected graph and since in every connected graph G, $C(G) = 1$, then $C\left(\mathfrak{R}_{\partial}(Z_{2p})\right) = 1$. But after removing the vertex p we get a disconnected graph and number of Components in this disconnected graph is

$\frac{p+1}{2}$ such that the components are show in the figure-3.4-

Figure 3.4: components

And the graph $\mathfrak{R}_{\partial}(Z_{2p})$ is not blook since by (theorem 3.1) it has a cut-vertex p but have subgraphs are block such that the subgraphs are show in the figure-3.4- are block subgraphs.

Then the graph $\mathfrak{R}_{\partial}(Z_{2p})$ have $\frac{p+1}{2}$ subgraphs that are block subgraph.

Remark:

The graph $\mathfrak{R}_{\partial}(Z_{3p})$ is connected graph and since in every connected graph G number of components is equal to 1 then number of components in the graph $\mathfrak{R}_{\partial}(Z_{3p})$ is equal to 1. But after removing the vertices p and 2p in $\mathfrak{R}_{\partial}(Z_{3p})$ we get a disconnected graph and number of components in this disconnected graph is $\frac{p+1}{2}$ such that the components are show in the figure-3.5- and figure-3.6-

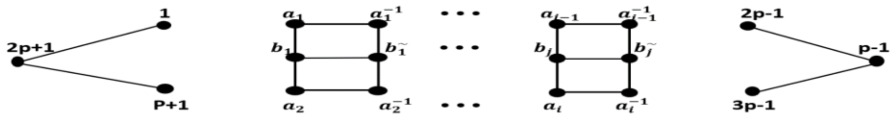

Figure 3.5

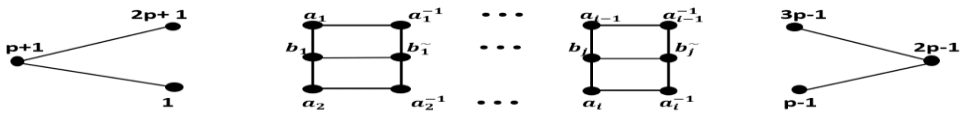

Figure 3.6

Theorem 3.7 $\delta(\mathfrak{R}_{\partial}(z_{2p})) = K(\mathfrak{R}_{\partial}(z_{2p})) = \lambda(\mathfrak{R}_{\partial}(z_{2p}))$.

Proof: In figure-2.4- it is clear the minimum vertex degree is 1 such that two vertices $p+1$ and $p-1$ have degree 1, then $\delta\left(R_d(z_{2p})\right) = 1$.

By theorem 3.1 the graph $\mathfrak{R}_{\partial}(z_{2p})$ is 1-connected graph then $K\left(\mathfrak{R}_{\partial}(z_{2p})\right) = 1$, and by theorem 3.3 the graph $\mathfrak{R}_{\partial}(z_{2p})$ is 1-edge connected, then $\lambda(\mathfrak{R}_{\partial}(z_{2p}))$ =1. So, we get the result $\delta(\mathfrak{R}_{\partial}(z_{2p})) = K(\mathfrak{R}_{\partial}(z_{2p})) = \lambda(\mathfrak{R}_{\partial}(z_{2p}))$.

Theorem 3.8 $\delta(\mathfrak{R}_{\partial}(z_{3p})) = K(\mathfrak{R}_{\partial}(z_{3p})) = \lambda(\mathfrak{R}_{\partial}(z_{3p}))$.

Proof: In figure-2.10- and figure-2.11- it is clear the minimum vertex degree is 2 such that in both cases the vertices $1, p+1.2p+1, p-1, 2p-1$ and $3p-1$ have degree 2. Then $\delta(\mathfrak{R}_{\partial}(z_{3p}))$ =2.

By theorem 3.2 the graph $\mathfrak{R}_{\partial}(z_{3p})$ is 2-connected graph then $K\left(\mathfrak{R}_{\partial}(z_{3p})\right) = 2$.

And by theorem 3.4 the graph $\mathfrak{R}_{\partial}(z_{3p})$ is 2-edge connected, then $\lambda\left(\mathfrak{R}_{\partial}(z_{3p})\right)$=2

So, we get the result $\delta(\mathfrak{R}_{\partial}(z_{3p})) = K(\mathfrak{R}_{\partial}(z_{3p})) = \lambda(\mathfrak{R}_{\partial}(z_{3p}))$.

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