



Measure Over Fuzzy P- Field and Some of its Properties

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ABSTRACT

In this paper, we introduce the concept of measure with respect to a fuzzy p-field. We define fuzzy measurable spaces, triple fuzzy measurable spaces, countable additivity, null fuzzy sets, and outer measures within this framework. We establish several propositions to demonstrate the properties and relationships among these concepts. We also examine the behavior of fuzzy measurable functions across various fuzzy fields and explore how measurability is preserved or lost under specific mappings.

Keywords: Fuzzy set, field, fuzzy field, σ -field, measurable space, fuzzy measurable function, π -system.

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قياس الحقل الضبابي P وبعض خصائصه

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الملخص

في هذا البحث، نقدم مفهوم القياس بالنسبة إلى الحقل الضبابي من النوع p. كما نعرّف الفضاءات القابلة للقياس ضبابياً، والفضاء الثلاثي القابل للقياس ضبابياً، وقابلية الإضافة المحدودة، والمجموعات الضبابية المنعدمة، والقياسات الخارجية ضمن هذا الإطار النظري. وتمّ طرح عدد من القضايا النظرية (Propositions) لإظهار الخصائص والعلاقات بين هذه المفاهيم. بالإضافة إلى ذلك، ندرس سلوك الدوال القابلة للقياس ضبابياً ضمن حقول ضبابية متعددة، ونستكشف كيف يمكن الحفاظ على خاصية القابلية للقياس أو فقدانها تحت تأثير تحويلات معينة.

1. INTRODUCTION

The notion of a fuzzy subset was introduced in ⁽¹⁾ to represent uncertainty. A fuzzy set $\tilde{K}$ is characterized by a membership function that assigns each element of the underlying set a value in the real interval $I = [0, 1]$. The collection of all fuzzy sets on $\tilde{K}$ is denoted by $P(\tilde{K})$. Subsequently, (2, 3) introduced related concepts in fuzzy set theory. The foundations of fuzzy control and the connections among fuzzy sets, fuzzy logic, and fuzzy control systems were discussed in ⁽⁴⁻⁶⁾. Furthermore, (7) examined difference and symmetric difference for intuitionistic fuzzy sets.

Set theory and its operations provide a basis for these developments ⁽⁸⁾, while (9) considers the concept of a ring in a neutrosophic setting. In measure theory, σ -algebras provide the framework for defining measures and studying their properties ⁽¹⁰⁾. Related concepts, including complete measures, measurable sets and functions, outer measures, integration, and extensions of measures to a σ -algebra, have been studied in ⁽¹¹⁻¹⁷⁾.

2. THE RESULTS

We start this section with the following definitions :

2-1. Measure space with respect to fuzzy p-field

In this section, we introduce fuzzy measurable space with respect to a fuzzy p-field and triple fuzzy measurable space with respect to a fuzzy p-field, countable additivity with respect to a fuzzy p-field, null fuzzy set in a fuzzy p-field, and some Propositions.

Definition (2.1.1): If $(Q, \tilde{K})$ is a fuzzy measurable space with respect to a fuzzy p-field. Then a set function $\tilde{\Psi} : \tilde{K} \rightarrow [0, \infty]$ is said to be a measure with respect to the fuzzy p-field $\tilde{K}$ if

1. $\tilde{\Psi}(\Phi) = 0$
2. $\tilde{\Psi}(\bigcup_{k=1}^{\infty} N_k) = \sum_{k=1}^{\infty} \tilde{\Psi}(N_k)$, for any $N_1, N_2, \dots \in \tilde{K}$ are disjoint.

Definition (2.1.2): If $(Q, \tilde{K})$ is a fuzzy measurable space over a fuzzy p-field and $\tilde{\Psi}$ a measure over a fuzzy p-field $\tilde{K}$, then a triple $(Q, \tilde{K}, \tilde{\Psi})$ is said to be a measure space over a fuzzy p-field.

Example (2.1.3): If $Q = \{x_1, x_2, \dots\}$ and $u_{N_i}(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$,

$\tilde{K} = \{\Phi, \tilde{X}, N_i = \{x_j, u_{N_j}(x_j)\}, \bigcup_i N_i \text{ and } j=1, 2, \dots\}$.
 $\tilde{X} = \{(x, 1) : \forall x \in Q\}$

We get that $(Q, \tilde{K})$ is a fuzzy measurable space over a fuzzy p-field. The set function $\tilde{\Psi} :$

$\tilde{K} \rightarrow [0, \infty]$ is : $\tilde{\Psi}(N_m) = \sum_{i=1}^{\infty} u_{N_m}(x_i)$, $m=1, 2, \dots$,
 $\tilde{\Psi}(\Phi) = 0$, and $\tilde{\Psi}(\bigcup_{k=1}^{\infty} N_k) = \sum_{i=1}^{\infty} u_{N_i}(x_i)$,
 $\tilde{\Psi}(\tilde{X}) = \infty$

$N_1 = \{(x_1, 1), (x_2, 0), (x_3, 0), \dots\}$

$N_2 = \{(x_1, 0), (x_2, 1), (x_3, 0), \dots\}$

.

$N_k = \{(x_1, 0), (x_2, 0), (x_3, 0), \dots, (x_k, 1), \dots\}$

Then $\tilde{\Psi}(\bigcup_{k=1}^{\infty} N_k) = 1+1+1 \dots = \infty$, $\tilde{\Psi}(N_k) = \sum_{i=1}^{\infty} u_{N_k}(x_i) = 1$, for all $N_k \in \tilde{K}$,

$k = 1, 2, \dots$ Hence, $\tilde{\Psi}(\bigcup_{k=1}^{\infty} N_k) = \sum \tilde{\Psi}(N_k)$, and $\tilde{\Psi}(\bigcap_k N_k) = 0$ for any $N_1, N_2, \dots \in \tilde{K}$. Then $\tilde{\Psi}$ is a measure over the fuzzy P-field.

Example (2.1.4): Let $Q = \{a_1, a_2\}$, and $\tilde{K} = \{\tilde{X}, \Phi, \{(a_1, 1), (a_2, 1)\}, \{(a_1, 0.2), (a_2, 0.6)\}, \{(a_1, 0.3), (a_2, 0.7)\}, \{(a_1, 0.4), (a_2, 0.5)\}\}$.

The pair $(Q, \tilde{K})$ is a fuzzy measurable space over a fuzzy p-field.

The function set $\tilde{\Psi} : \tilde{K} \rightarrow [0, \infty]$ is $\tilde{\Psi}(N) = \begin{cases} 1, & \text{if } N = \tilde{X} \\ 1, & \text{other wise} \end{cases}$. Consider

$N_1 = \{(a_1, 1), (a_2, 1)\}$, $N_2 = \{(a_1, 0.2), (a_2, 0.6)\}$. Then $N_1 \cap N_2 = \{(a_1, 0.2), (a_2, 0.6)\}$ and

$N_1 \cup N_2 = \tilde{X}$, $\tilde{\Psi}(N_1) = \tilde{\Psi}(N_2) = 1$. So, we have $\tilde{\Psi}(N_1 \cup N_2) = \tilde{\Psi}(\tilde{X}) = 1$, and $\sum_{k=1}^2 \tilde{\Psi}(N_k) = 1+1=2$. Hence,

$\sum_{k=1}^2 \tilde{\Psi}(N_k) \neq \tilde{\Psi}(N_1 \cup N_2)$. Then $\tilde{\Psi}$ is not measured over the fuzzy p-field.

Definition (2.1.5): If $(Q, \tilde{K})$ is a fuzzy measurable space over a fuzzy p-field. The function set $\tilde{\Psi} : \tilde{K} \rightarrow [-\infty, \infty]$ is called countably additive over the fuzzy p-field $\tilde{K}$ if

1. $\tilde{\Psi}(\Phi) = 0$
2. $\tilde{\Psi}(\bigcup_k N_k) = \sum \tilde{\Psi}(N_k)$, for any $N_1, N_2, \dots \in \tilde{K}$ are disjoint.

Example (2.1.6): Let $Q = \{x_1, x_2\}$, and $\mathcal{K} = \{\tilde{X}, \Phi, \{(x_1, 1), (x_2, 1)\}, \{(x_1, 0.2), (x_2, 0.1)\}\}$

The pair $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field.

The function set $\tilde{\Psi} : \mathcal{K} \rightarrow [-\infty, \infty]$ is

$$\tilde{\Psi}(N) = \begin{cases} -6, & \text{if } N = \tilde{X} \\ 0, & \text{otherwise} \end{cases}, \text{ Consider}$$

$N_1 = \{(x_1, 1), (x_2, 1)\}$, $N_2 = \{(x_1, 0.2), (x_2, 0.1)\}$. Then $N_1 \widetilde{\cap} N_2 = \{(x_1, 0.2), (x_2, 0.1)\}$, and

$N_1 \widetilde{\cup} N_2 = \tilde{X}$. Then $\tilde{\Psi}(N_1 \widetilde{\cup} N_2) = \tilde{\Psi}(\tilde{X}) = -6$, $\tilde{\Psi}(N_1) = -6$, $\tilde{\Psi}(N_2) = 0$. Hence,

$\sum_{k=1}^2 \tilde{\Psi}(N_k) = \tilde{\Psi}(N_1 \widetilde{\cup} N_2) = -6$. Then $\tilde{\Psi}$ is a countable addition over a fuzzy p-field.

Example (2.1.7): If $Q = \{w_1, w_2\}$, and $\mathcal{K} = \{\tilde{X}, \Phi, \{(w_1, 0.1), (w_2, 0.3)\}, \{(w_1, 1), (w_2, 1)\}\}$

The pair $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field. The function set

$$\tilde{\Psi} : \mathcal{K} \rightarrow [-\infty, \infty] \text{ is } \tilde{\Psi}(N) = \begin{cases} -4, & \text{if } N = \tilde{X} \\ 0, & \text{if } N = \Phi \\ -1, & \text{otherwise} \end{cases},$$

Consider

$N_1 = \{(w_1, 0.1), (w_2, 0.3)\}$, and $N_2 = \{(w_1, 1), (w_2, 1)\}$. Then $N_1 \widetilde{\cap} N_2 = \{(w_1, 0.1), (w_2, 0.3)\}$ and

$N_1 \widetilde{\cup} N_2 = \tilde{X}$. We get $\tilde{\Psi}(N_1 \widetilde{\cup} N_2) = \tilde{\Psi}(\tilde{X}) = -4$, and $\tilde{\Psi}(N_1) = -4$, $\tilde{\Psi}(N_2) = -1$. Hence, $\sum_{k=1}^2 \tilde{\Psi}(N_k) = -5 \neq \tilde{\Psi}(N_1 \widetilde{\cup} N_2)$. Then $\tilde{\Psi}$ is not countable additive over a fuzzy p-field.

Remark (2.1.8): Let $(Q, \mathcal{K})$ be a fuzzy measurable space over a fuzzy p-field. Then the measure on $\mathcal{K}$ is a countably additive measure on $\mathcal{K}$, but the inverse is not true.

The inverse is not true; for example,

Example (2.1.9): Let $Q = \{w_1, w_2\}$, and $\mathcal{K} = \{\tilde{X}, \Phi, \{(w_1, 1), (w_2, 1)\}, \{(w_1, 0.3), (w_2, 0.7)\}\}$

The pair $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field.

The function set $\tilde{\Psi} : \mathcal{K} \rightarrow [-\infty, \infty]$ is :

$$\tilde{\Psi}(N) = \begin{cases} -2, & \text{if } N = \tilde{X} \\ 0, & \text{otherwise} \end{cases}, \text{ Consider}$$

$N_1 = \{(w_1, 1), (w_2, 1)\}$, $N_2 = \{(w_1, 0.3), (w_2, 0.7)\}$. Then $N_1 \widetilde{\cap} N_2 = \{(w_1, 0.3), (w_2, 0.7)\}$.

And $N_1 \widetilde{\cup} N_2 = \tilde{X}$. We get $\tilde{\Psi}(N_1 \widetilde{\cup} N_2) = \tilde{\Psi}(\tilde{X}) = -2$, and $\tilde{\Psi}(N_1) = -2$, $\tilde{\Psi}(N_2) = 0$. Hence, $\sum_{k=1}^2 \tilde{\Psi}(N_k) = \tilde{\Psi}(N_1 \widetilde{\cup} N_2) = -2$. Then $\tilde{\Psi}$ is a countable additivity over the fuzzy p-field. It is not measurable over the fuzzy p-field because $\tilde{\Psi}$ is a measure

(i.e. $\tilde{\Psi} : \mathcal{K} \rightarrow [0, \infty]$, then $(-3) \notin [0, \infty]$). So, we get $\tilde{\Psi}(N) \geq 0$, $\forall N \in \mathcal{K}$, however $\tilde{\Psi}(\{(w_1, 0.3), (w_2, 0.7)\}) = -3$, this is opposition.

Proposition (2.1.10): If the pair $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field, assume $\tilde{\Psi}_1, \tilde{\Psi}_2$ are measures on $\mathcal{K}$, such that $\tilde{\Psi}_1(A)$ or $\tilde{\Psi}_2(A)$ finite $\forall A \in \mathcal{K}$. The $\tilde{\Psi}_1 - \tilde{\Psi}_2$:

$\mathcal{K} \rightarrow [-\infty, \infty]$ is : $[\tilde{\Psi}_1 - \tilde{\Psi}_2](N) = \tilde{\Psi}_1(N) - \tilde{\Psi}_2(N)$, $\forall N \in \mathcal{K}$, then $\tilde{\Psi}_1 - \tilde{\Psi}_2$ is a countable additive over $\mathcal{K}$.

Proof:

If $\tilde{\Psi}_1, \tilde{\Psi}_2$ are measures over a fuzzy p-field. There $\tilde{\Psi}_1(\Phi) = \tilde{\Psi}_2(\Phi) = 0$ and hence $[\tilde{\Psi}_1 - \tilde{\Psi}_2](\Phi) = \tilde{\Psi}_1(\Phi) - \tilde{\Psi}_2(\Phi) = 0$. If $N_1, N_2, \dots$ form a countable infinite collection of pairwise disjoint fuzzy sets in $\mathcal{K}$. Such that $\tilde{\Psi}_i(\widetilde{\cup} N_n) = \sum_n \tilde{\Psi}_i(N_n)$, $i=1, 2$. So, we have $[\tilde{\Psi}_1 - \tilde{\Psi}_2](\widetilde{\cup} N_n) = \tilde{\Psi}_1(\widetilde{\cup} N_n) - \tilde{\Psi}_2(\widetilde{\cup} N_n) = \sum_n \tilde{\Psi}_1(N_n) - \sum_n \tilde{\Psi}_2(N_n) = \sum_n [\tilde{\Psi}_1(N_n) - \tilde{\Psi}_2(N_n)] = \sum_n [\tilde{\Psi}_1 - \tilde{\Psi}_2](N_n)$.

Then $\tilde{\Psi}_1 - \tilde{\Psi}_2$ is a countable addit over $\mathcal{K}$.

Definition (2.1.11): If $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field, let $\tilde{\Psi}$ is a measure over $\mathcal{K}$. The fuzzy set N in $\mathcal{K}$ is called an empty fuzzy set if

• $\tilde{\Psi}(N) = 0$.

Definition (2.1.12): If $Q \neq \Phi$, $\mathcal{K} = P(Q)$. Then $(Q, \mathcal{K})$ is a fuzzy measurable space over a fuzzy p-field. Defined as $\tilde{\Psi} : \mathcal{K} \rightarrow [0, \infty]$ such that : $\tilde{\Psi}(N) = 0$, $\forall N \in \mathcal{K}$.

2-2. Fuzzy measurable function

In this section, we introduce fuzzy measurable functions over the fuzzy p-field, fuzzy measurable functions over fuzzy β -field, fuzzy measurable function over fuzzy $\beta - \sigma$ -field, fuzzy measurable

unction over fuzzyrurable function over fuzzy η -field.

Definition (2.2.1): Let $(Q_1, \mathcal{K}_1)$, $(Q_2, \mathcal{K}_2)$ are two fuzzy measurable spaces over fuzzy p - fields.

Then $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over fuzzy p - fields if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Example (2.2.2): Let $Q_1 = \{a, b, c\}$, $Q_2 = \{m, n, o\}$, $f(N) = (x, 1 - \mu_N)$

$\mathcal{K}_1$

$\mathcal{K}_1 = \{\Phi, \check{X}_1, \{(a, 0.6), (b, 0.7), (c, 1)\}, \{(a, 0.6), (b, 0.8), (c, 0)\}\}$. And

$\mathcal{K}_2 = \{\Phi, \check{X}_2, \{(m, 0.4), (n, 0.2), (o, 1)\}, \{(m, 0.4), (n, 0.3), (o, 0)\}\}$. Put $N_1 = \{(a, 0.6), (b, 0.7), (c, 1)\} \in \mathcal{K}_1$. $N_2 = \{(a, 0.6), (b, 0.8), (c, 0)\} \in \mathcal{K}_1$.

define $f(N_1) = \{(m, 0.4), (n, 0.3), (o, 0)\} \in \mathcal{K}_2$, and $f(N_2) = \{(m, 0.4), (n, 0.2), (o, 1)\} \in \mathcal{K}_2$.

We get f a fuzzy measurable function over a fuzzy p - field.

Example (2.2.3): Let $Q_1 = \{n, h\}$, $Q_2 = \{j, m\}$, $\mathcal{K}_1 = \{\Phi, \check{X}_1, \{(n, 0.6), (h, 0.2)\}, \{(n, 0.4), (h, 1)\}\}$,

$\mathcal{K}_2 = \{\Phi, \check{X}_2, \{(j, 0.3), (m, 0.1)\}, \{(j, 0.2), (m, 0.5)\}\}$. $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$. But, $N_1 = \{(n, 0.6), (h, 0.2)\}$, $N_2 = \{(n, 0.4), (h, 1)\}$. Define $f(N_i) = (x, \frac{\mu_{N_i}}{0.2}) = \{\Phi, \check{X}_2, \{(j, 0.3), (m, 0.1)\}, \{(j, 0.2), (m, 0.5)\}\}$.

Then f is not a fuzzy measurable function over a fuzzy p - field.

Definition (2.2.4): Let $(Q_1, \mathcal{K}_1)$, $(Q_2, \mathcal{K}_2)$ be two fuzzy measurable spaces over a fuzzy β - field. Then $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over the fuzzy β - field, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Example (2.2.5): Let $Q_1 = \{a, b\}$, $Q_2 = \{t, s\}$, $f(N) = (x, |\mu_N - 1|)$

$\mathcal{K}_1 = \{\Phi, \{(a, 0.6), (b, 0.7)\}, \{(a, 0.9), (b, 0.8)\}, \text{and}$

$\mathcal{K}_2 = \{\Phi, \{(t, 0.4), (s, 0.3)\}, \{(t, 0.1), (s, 0.2)\}\}$.

Put $N_1 = \{(a, 0.6), (b, 0.7)\}$, $N_2 = \{(a, 0.9), (b, 0.8)\}$. defined by $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, $f(N_1) = \{(t, 0.4), (s, 0.3)\} \in \mathcal{K}_2$, and $f(N_2) = \{(t, 0.1), (s, 0.2)\} \in \mathcal{K}_2$.

We get f a fuzzy measurable function over fuzzy. β - field.

Proposition (2.2.6): If $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over p -field, then f is a fuzzy measurable function over the fuzzy β -field.

Proof :

Since f is a fuzzy measurable function over the fuzzy p - field, then $(Q_1, \mathcal{K}_1)$ and $(Q_2, \mathcal{K}_2)$ are fuzzy measurable spaces over the fuzzy p - field, and since every fuzzy measurable space over the fuzzy p - field is a fuzzy measurable space over the fuzzy. β - field, we get, $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$. Then we have $f(\Phi) = \Phi \in \mathcal{K}_2$. The inverse of the over property is not essentially true, as shown in the next example.

Example (2.2.7): If $Q_1 = \{l, b\}$, $Q_2 = \{h, i\}$, $f(N) = (x, \mu_N - 0.1)$, $\mathcal{K}_1 = \{\Phi, \{(l, 0.7), (b, 0.6)\}, \{(l, 0.8), (b, 0.9)\}\}$. And

$\mathcal{K}_2 = \{\Phi, \{(h, 0.6), (i, 0.5)\}, \{(h, 0.7), (i, 0.8)\}\}$. $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Then f is a fuzzy measurable function over a fuzzy β - field. But it is not a fuzzy measurable function over the fuzzy p - field because $f(\check{X}_1) = \check{X}_2$ not exist.

Definition (2.2.8): Let $(Q_1, \mathcal{K}_1)$, $(Q_2, \mathcal{K}_2)$ be two fuzzy measurable spaces over fuzzy σ - β - field. Then $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over the fuzzy σ - β - field, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Example (2.2.9): Let $Q_1 = \{a, b\}$, $Q_2 = \{h, i\}$, $f(N_i) = (a_i, \frac{\mu_{N_i}}{2})$

$\mathcal{K}_1 = \{\Phi, \check{X}_1, \{(a, 0.2), (b, 0.6)\}, \{(a, 0.4), (b, 0.8)\}, \text{and}$

$\mathcal{K}_2 = \{\Phi, \check{X}_2, \{(h, 0.1), (i, 0.3)\}, \{(h, 0.2), (i, 0.4)\}\}$.

Put, $N_1 = \{(a, 0.2), (b, 0.6)\}$ $N_2 = \{(a, 0.4), (b, 0.8)\}$, define $f(N_1) = \{(h, 0.1), (i, 0.3)\} \in \mathcal{K}_2$, and $f(N_2) = \{(h, 0.2), (i, 0.4)\} \in \mathcal{K}_2$. We get that f is a fuzzy measurable function over the fuzzy σ - β - field.

Proposition (2.2.10) : If $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ a fuzzy measurable function over p -field, then f is a fuzzy measurable function over the fuzzy σ - β - field.

Proof :

Let f be a fuzzy measurable function over a fuzzy p - field. Then the fuzzy measurable function over a fuzzy p - field is defined by: $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$. Then we have

$f(\Phi) = \Phi \in \mathcal{K}_2$, $f(\tilde{X}_2) = \tilde{X}_1 \in \mathcal{K}_2$. Then f is a measurable function over the fuzzy $\sigma - \beta$ - field.

The inverse of the over property is not necessarily true; for example, next.

Example (2.2.11): Let $Q_1 = \{r, s\}$, $Q_2 = \{h, j\}$, $f(N) = (x, \left| \frac{1}{3\mu_{Ni}} - 1 \right|)$
 $\mathcal{K}_1 = \{\Phi, \tilde{X}_1, \{(r, 0.3), (s, 0.6)\}, \{(r, 0.4), (s, 0.5)\}, \{(r, 0.3), (s, 0.4)\}\}$, and $\mathcal{K}_2 = \{\Phi, \tilde{X}_2, \{(h, 0.1), (j, 0.4)\}, \{(h, 0.1), (j, 0.3)\}, \{(h, 0.1), (j, 0.1)\}\}$. $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Then f is a fuzzy measurable function over fuzzy $\sigma - \beta$ - field.

But is not a fuzzy measurable function over fuzzy p - field because $f(\max \mu_{N_n}) = \max f(\mu_{N_n}) = \{(h, 0.4), (j, 0.6)\} \notin \mathcal{K}_1$.

Definition (2.2.12): Let $(Q_1, \mathcal{K}_1), (Q_2, \mathcal{K}_2)$ be two fuzzy measurable spaces over a fuzzy η - field.

Then the $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over a fuzzy η - field if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Example (2.2.13): Let $Q_1 = \{a, b\}$, $Q_2 = \{m, n\}$, $f(N) = (x, 0.1\mu_N)$
 $\mathcal{K}_1 = \{\Phi, \tilde{X}_1, \{(a, 0.5), (b, 0.9)\}, \{(a, 1), (b, 1)\}, \text{and } \mathcal{K}_2 = \{\Phi, \tilde{X}_2, \{(m, 0.05), (n, 0.09)\}, \{(m, 0.1), (n, 0.1)\}\}$.
 $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$. put $N_1 = \{(a, 0.5), (b, 0.9)\}$, $N_2 = \{(a, 1), (b, 1)\}$. and $f(N_1) = \{(m, 0.05), (n, 0.09)\} \in \mathcal{K}_2$, and $f(N_2) = \{(m, 0.1), (n, 0.1)\} \in \mathcal{K}_2$.

Then f is a fuzzy measurable function over fuzzy η - field.

Proposition (2.2.14): If $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over p - field, such that f is a fuzzy measurable function over a fuzzy η - field.

Proof :

Let f be a fuzzy measurable function over a fuzzy p - field. Then the fuzzy measurable function over a fuzzy p - field is defined by: $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$. Then we have $f(\Phi) = \Phi \in \mathcal{K}_2$, $f(\tilde{X}_2) = \tilde{X}_1 \in \mathcal{K}_1$. We get that f is a measurable function over the fuzzy η - field.

The inverse of the over property is not necessarily true; for example, next.

Example (2.2.15): Assume $\mathcal{K} = \{\Phi, \tilde{X}, \{(a, 0.5), (b, 0)\}, \{(a, 0), (b, 0.4)\}\}$.

Then $(Q, \mathcal{K})$ is a fuzzy measurable function over η - field. Then $f: (Q, \mathcal{K}) \rightarrow (Q, \mathcal{K})$, and is defined $f(N_i) = (a_i, \mu_{N_i})$, $f(N_1) = \{(a, 0.5), (b, 0)\}$, $f(N_2) = \{(a, 0), (b, 0.4)\}$, $f(\Phi) = \Phi$.

$f(\tilde{X}_1) = \tilde{X}_2$. Then $f(N_i) = \{\Phi, \tilde{X}, \{(a, 0.5), (b, 0)\}, \{(a, 0), (b, 0.4)\}\}$ is a fuzzy measurable function over the fuzzy η - field but not a fuzzy measurable function over the fuzzy p - field because

$\cup N_i = \{(a, 0.5), (b, 0.4)\} \notin f(N_i)$.

Definition (2.2.16): Let $(Q_1, \mathcal{K}_1), (Q_2, \mathcal{K}_2)$ be two fuzzy measurable spaces over fuzzy π - system. Then $f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$ is a fuzzy measurable function over fuzzy π - system, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$.

Example (2.2.17): Let $Q_1 = \{j, k\}$, $Q_2 = \{c, d\}$, $f(N) = (x, \frac{\mu_N}{3})$

$\mathcal{K}_1 = \{\Phi, \{(j, 0.3), (k, 0.6)\}, \{(j, 0.6), (k, 0.9)\}, \text{and } \mathcal{K}_2 = \{\Phi, \{(c, 0.1), (d, 0.2)\}, \{(c, 0.2), (d, 0.3)\}\}$.

$f: (Q_1, \mathcal{K}_1) \rightarrow (Q_2, \mathcal{K}_2)$, if $\forall N \in \mathcal{K}_1, f(N) \in \mathcal{K}_2$. put $N_1 = \{(j, 0.3), (k, 0.6)\}$, $N_2 = \{(j, 0.6), (k, 0.9)\}$. And

$f(N_1) = \{(c, 0.1), (d, 0.2)\} \in \mathcal{K}_2$, and $f(N_2) = \{(c, 0.2), (d, 0.3)\} \in \mathcal{K}_2$.

Then f is a fuzzy measurable function over a fuzzy π - system.

Proposition (2.2.18): Every fuzzy measurable function over a fuzzy p - field is a fuzzy measurable function over a fuzzy π - system.

Proof :

Assume that f is a fuzzy measurable function over a fuzzy p - field. Let $\mathcal{K}_2$ be elements $\tilde{K}_i \in \mathcal{K}_2$. Put $\tilde{K}_k = \tilde{X}_i, \forall k \geq 2$, then $\bigcap_{i=1}^{\infty} \tilde{K}_i \in \mathcal{K}_2$, but $\bigcap_{i=1}^{\infty} \tilde{K}_i = \tilde{K}_1 \cap \tilde{K}_2 \cap \tilde{X} \cap \dots = \tilde{K}_1 \cap \tilde{K}_2$, thus $\tilde{K}_1 \cap \tilde{K}_2 \in \mathcal{K}_2$. So, $\mathcal{K}_2$ is a fuzzy measurable function over a fuzzy π - system. The inverse of the over property is not necessarily true; for example, the following.

Example (2.2.19): Let $Q_1 = \{a, s\}$, $Q_2 = \{h, i\}$, $f(N) = (x, \mu_N - 1)$.

$K_1 = \{ \{(a, 0.7), (s, 0.6) \}, \{(a, 0.8), (s, 0.9) \}, \text{and}$

$K_2 = \{ \{(h, 0.3), (i, 0.4) \}, \{(h, 0.2), (i, 0.1) \} \}$

$f : (Q_1, K_1) \rightarrow (Q_2, K_2)$, if $\forall N \in K_1$, $f(N) \in K_2$.

We get f as a fuzzy measurable function over a fuzzy π -system. But it is not a fuzzy measurable function over a fuzzy p -field because

$f(\tilde{X}_1) = \tilde{X}_2$ not exist.

Definition (2.2.20): If (Q, K) is a fuzzy measurable space over a fuzzy p -field.

The function set $\tilde{\Psi} : K \rightarrow [0, \infty]$ is called an outer measure on the fuzzy p -field K if

1. $\tilde{\Psi}(\Phi) = 0$
2. $\tilde{\Psi}(\bigcup_{n=1}^{\infty} N_n) \leq \sum_{n=1}^{\infty} \tilde{\Psi}(N_n)$, for any $N_1, N_2, \dots \in K$ are disjoint.
3. If $N, \hat{A} \in K$, and $\hat{A} \leq N$, such that $\tilde{\Psi}(\hat{A}) \leq \tilde{\Psi}(N)$.

Example (2.2.21): If $Q = \{g, b\}$, and $K = \{ \tilde{X}, \Phi, \{(g, 0.2), (b, 0.6) \}, \{(g, 0.3), (b, 0.7) \} \}$.

The pair (Q, K) is a fuzzy measurable space over a fuzzy p -field.

The function set $\tilde{\Psi} : K \rightarrow [0, \infty]$ is :

$\tilde{\Psi}(N) = \begin{cases} 6, & \text{if } N = \{(g, 0.3), (b, 0.7) \} \\ 3, & \text{other wise} \end{cases}$, Consider

$N_1 = \{(g, 0.2), (b, 0.6) \}$, $N_2 = \{(g, 0.3), (b, 0.7) \}$. Then

$N_1 \tilde{\cup} N_2 = \{(g, 0.3), (b, 0.7) \}$. So, we have

$\tilde{\Psi}(N_1 \tilde{\cup} N_2) = \tilde{\Psi}(\{(g, 0.3), (b, 0.7) \}) = 6$, and $\tilde{\Psi}(N_1)$

$= 6$, $\tilde{\Psi}(N_2) = 3$, $\sum_{k=1}^2 \tilde{\Psi}(N_k) = 3 + 6 = 9$ Hence,

$\tilde{\Psi}(N_1 \tilde{\cup} N_2) \leq \sum_{k=1}^2 \tilde{\Psi}(N_k)$ and $\tilde{\Psi}(N_2) \leq \tilde{\Psi}(N_1)$.

Then $\tilde{\Psi}$ is the outer measure over the fuzzy p -field.

Example (2.2.22): If $Q = \{a, b\}$, and $K = \{ \tilde{X}, \Phi, \{(a, 0), (b, 0.1) \}, \{(a, 0.4), (b, 0.8) \} \}$

The pair (Q, K) is a fuzzy measurable space over a fuzzy p -field.

The function set $\tilde{\Psi} : K \rightarrow [0, \infty]$

is: $\tilde{\Psi}(N) = \begin{cases} 4, & \text{if } N = \{(a, 0.4), (b, 0.8) \} \\ 8, & \text{other wise} \end{cases}$, Consider

$N_1 = \{(a, 0), (b, 0.1) \}$, $N_2 = \{(a, 0.4), (b, 0.8) \}$. Then

$N_1 \tilde{\cup} N_2 = \{(a, 0.4), (b, 0.8) \}$. We get $\tilde{\Psi}(N_1 \tilde{\cup} N_2)$

$= \tilde{\Psi}(\{(a, 0.4), (b, 0.8) \}) = 4$, and $\tilde{\Psi}(N_1) = 4$, $\tilde{\Psi}(N_2) = 8$,

$\sum_{k=1}^2 \tilde{\Psi}(N_k) = 4 + 8 = 12$.

Hence, $\tilde{\Psi}(N_1 \tilde{\cup} N_2) \leq \sum_{k=1}^2 \tilde{\Psi}(N_k)$ Then $\tilde{\Psi}$ is not an

outer measure over the fuzzy p -field because $\tilde{\Psi}(N_2) \not\leq \tilde{\Psi}(N_1)$.

3. CONCLUSION

This study has developed a comprehensive framework for defining measures over fuzzy p -fields. We showed that countable additivity may hold without satisfying all the properties of a measure, and we provided several examples to illustrate these distinctions. We also demonstrated that fuzzy measurable functions over fuzzy p -fields inherently satisfy measurability over fuzzy σ -fields and π -systems, although the converse may not always be true. These results provide a foundation for further exploration of fuzzy integration and applications in fuzzy control systems.

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