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Continuous Threshold Harvesting Intermediate Predator in Food Chain Model

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Abstract
This paper presents a three-species food chain system, which consists of intermediate predator population that depends only on prey population, and top predator population that depends only on intermediate predator population. We study this model when the intermediate predator exposed to the risk of harvest.
We studied the bounded solutions and equilibrium points with its conditions. Also the stability for each equilibrium points was studied. We determine the invariant region, in this region all population are survive and continuous harvesting. At last, we describe some results in numerical simulation.

Key words: Food chain, equilibrium point, harvesting, invariant region.

1. Introduction
The study of interaction between populations of various species is an active research area in theoretical ecology. One of the earliest and important of such studies is the interactions between predators and prey. The model describing such interactions was first investigated by Lotka and later independently by Volterra. The model, which came to be known as the Lotka-Volterra system of equations, were further developed and extended by many authors.
Predator–prey models are of great significance in mathematics and biology. Predator–prey models have attracted a lot of attention in recent years because they represent mathematical equations that deal with important ecological problems such as the spread of diseases and irregularity in harvesting that can lead to the extinction of species. Researchers have investigated the various forms of predator–prey models, Gakkhar and Naji [6] , Naji and Balasim [12] , and Upadhyay and Naji [13] studied the dynamics of several forms of predator–prey models for different functional responses, Wuhaib and Hasan [14] and Chauvet, Paullet et al. [3] studied the dynamics of food chains and fixed surfaces. Xiao, Li et al. [15] studied the dynamics of a predator–prey model with an infected prey living in a habitat accessible to the predator. Kumar Kar [11] analyzed the effects of constant and random refuges with harvests to show the effect of refuges on stability. Butler, Hsu et al. [2], Hwang [10] and Hesaaraki and Moghadas [8] studied the local stability, global stability and limit cycle of predator–prey models. Freedman and Waltman [5] developed hypotheses on the persistence. This idea was further explored by others Heathcoteetal [9], Haque et al [7] Another important extension was the incorporation of harvesting in the model as carried out by Duby [4], and Xiao [16].
Most predator prey models with harvesting consider either constant or linear harvesting functions and assumed that harvesting starts at. Jonathan Bohn and Jorge Rebaza [1] study the continuous threshold prey harvesting dependent on size of prey in predator prey model. Classically, the harvesting function is defined as

$$H(y)=\begin{cases}0 & \text{if } y<T,\\ h & \text{if } y\geq T.\end{cases} \quad (1)$$

Where y is the prey or predator that will be harvested, T is the threshold while h is the rate of harvesting. In this paper, we use two continuous threshold policy (*CTP*) harvesting functions on the prey Bohn [1]. One of them has the form

$$H(y)=\begin{cases}0 & \text{if } y<T,\\ \dfrac{h(y-T)}{h+y-T} & \text{if } y\geq T.\end{cases} \quad (2)$$

We study the continuous threshold intermediate predator in food chain model. We assume the intermediate predator will only interact with the prey and the top predator will only interact with is on the intermediate predator. The intermediate predator exposed to the risk of harvest.

2 .The model
We now introduce several variations of the model.

2.1The model without harvesting
The model can be written as

$$\begin{cases}x'=x(1-x)-k_1xy\\ y'=k_2xy-b_1\dfrac{yz}{1+y}\\ z'=b_2\dfrac{yz}{1+y}-\gamma z\end{cases} \quad (3)$$

Where x , y and z are prey, intermediate predator and top predator respectively, k_1, k_2 the contact between prey and intermediate predator, b_1, b_2 the depletion and γ the death rate of top predator. The term $yz/(1+y)$ is the Holling functional response type II.

Theorem 1. If $\gamma \geq b_2$ then $lim\ z(t)=0\ as\ t\to\infty$

Proof . Suppose that $b_2<\gamma$ then $\frac{dz}{dt}<0$, therefore, $lim\ z(t)$ exist and non negative.

To show that $lim\ z(t)=0\ as\ t\to\infty$ suppose there exist a positive constant q such that

$lim\ z(t) = q\ as\ t \to \infty$.

We take a small positive value ε such that $q - \varepsilon < z(t) < q + \varepsilon$ as $t > t_0$ and take the positive constant y_{max} such that $y(t) < y_{max}$ when $t > t_0$. From the third equation of system (3) we get

$$\frac{dz}{dt} = -\gamma z + b_2 \frac{yz}{1+y}$$

i.e.

$$\frac{1}{z}\frac{dz}{dt} = -\gamma + b_2 \frac{y}{1+y}$$

Then

$$z(t) = z(t_0)\exp\left(\int_{t_0}^{t}\left(\frac{b_2 y(s)}{1+y(s)} - \gamma\right)ds\right)$$

Since $z(s) > p - \varepsilon$ when $t > t_0$, we have

$$z(t) \le z(t_0)\exp\left(\int_{t_0}^{t}\left(\frac{b_2 y(s)}{1+y(s)} - \gamma\right)ds\right)$$

$$z(t) \le z(t_0)\exp\left(-\frac{(b_2-\gamma)y_{max} - \gamma}{1+y_{max}}\right)(t-t_0) \to 0\ as\ t \to \infty$$

This is a contradiction with the hypothesis, thus $lim\ z(t) = 0\ as\ t \to \infty$ □

Theorem 2 The system (3) cannot have periodic orbit.

Proof. We use Dulac's Criterion to show the nonexistence of periodic orbit.

We choose the multiplier $g(x,y) = \frac{1}{xy}$ and consider the positive quadrant of the xy -plane**.**

Let $g_1(x,y) = x(1-x) - k_1 xy$ and $g_2(x,y) = k_2 xy$

Consider the divergence

$$\Delta(x,y) = \frac{\partial(g_1 g)}{\partial x} + \frac{\partial(g_2 g)}{\partial y}$$

We get $\Delta(x,y) = \frac{\partial}{\partial x}\left(\frac{1-x}{y} - k_1\right) + \frac{\partial}{\partial y}(k_2)$

i.e.

$$\Delta(x,y) = -\frac{1}{y}$$

Which is clearly sign-definite in the region considered **.** □

The model system (3) has the following equilibriums points:

- The trivial equilibrium point $p_0(0,0,0)$ always exists.
- The equilibrium point $p_1(1,0,0)$ exists on the boundary of the first octant.
- The nontrivial equilibrium point $p_2(x_2, y_2, z_2)$, where

$$x_2 = 1 - k_1 y_2, y_2 = \frac{\gamma}{b_2 - \gamma}, \quad z_2 = \frac{b_2 k_2 x_2 y_2}{b_1 \gamma}$$

Exists under the condition $(1 > x_2)$.

The variational matrix of system (3) is given by

$$J = \begin{bmatrix} 1-2x-k_1 y & -k_1 x & 0 \\ k_2 y & k_2 x - b_1 \frac{z}{(1+y)^2} & -b_1 \frac{y}{1+y} \\ 0 & b_2 \frac{z}{(1+y)^2} & b_2 \frac{y}{(1+y)} - \gamma \end{bmatrix}$$

From the linear stability analysis about the above equilibrium points, we have the following:

- The eigenvalues of p_0 are $\lambda_1 = 1 > 0, \lambda_2 = 0$ and $\lambda_3 = -\gamma < 0$, so it is an unstable manifold along x -direction while a stable manifold along z -direction. Hence p_0 a saddle point.
- The eigenvalues of p_1 are $\lambda_1 = -1 < 0$, $\lambda_2 = k_2 > 0$ and $\lambda_3 = -\gamma < 0$, so it is locally asymptotically stable.
- Next, for the positive equilibrium p_2 , the variational matrix is

$$J_{p_2} = \begin{bmatrix} -x_2 & -k_1 x_2 & 0 \\ k_2 y_2 & k_2 x_2 - b_1 \frac{z_2}{(1+y_2)^2} & -\frac{\gamma b_1}{b_2} \\ 0 & b_2 \frac{z_2}{(1+y_2)^2} & 0 \end{bmatrix}$$

The characteristic equation of this point is $\lambda^3 + A\lambda^2 + B\lambda + C = 0$

Where

$$A_2 = (1-k_2)x_2 + b_1 \frac{z}{(1+y)^2},\ A_2 > 0 \text{ if } 1 > k_2$$

$$B_2 = -x_2\left(k_2 x_2 - b_1 \frac{z_2}{(1+y_2)^2}\right) + \frac{\gamma b_1 z_2}{(1+y_2)^2} + k_1 k_2 x_2 y_2$$

And

$$C_2 = \frac{b_1 \gamma x_2 z_2}{(1+y_2)^2} > 0$$

By Routh-Hurwitz criterion this point is stable is $A_2 B_2 - C_2 > 0$

2.2 The model with harvesting

We use the continuous threshold policy (CTP) harvesting function on the susceptible prey, which has the form in equation (2). Here T is the threshold value and h the rate of harvest. When $y \ge T$ the harvesting starts and the model becomes

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$$\begin{cases} x' = x\,(1-x) - k_1 xy \\ y' = k_2 xy - b_1 \dfrac{yz}{1+y} - H\,(y) \\ z' = b_2 \dfrac{yz}{1+y} - \gamma z \end{cases} \qquad (4)$$

2.2.1 Bound on the Solution

Theorem 3. The solution to system of equations (4) is bounded.

Proof. Same proof in theorem 1.

2.2.2. Equilibrium and Stability

The model in system (4) has the following equilibriums points:

- The trivial equilibrium point $P_3(0,0,0)$ always exists.
- The equilibrium point $P_4(1,0,0)$ exists on the boundary of the first octant.
- The nontrivial equilibrium point $P_5(x_5, y_5, z_5)$, where

$$x_5 = 1 - k_1 y_5,\ y_5 = \frac{\gamma}{b_2 - \gamma},\ z_5 = \frac{b_2}{\gamma b_1}\left[k_2 x_5 y_5 - H\,(y)\right]$$

Exists under the conditions $1 > k_1 y_5$ $b_2 > \gamma$, and $k_2 x_5 y_5 > H\,(y)$.

Accordingly, the linear stability analysis about the above equilibrium points gives the following:

- The eigenvalues of P_3 are $\lambda_1 = 1$, $\lambda_2 = -\dfrac{h^2}{(h-T)^2}$ and $\lambda_3 = -\gamma < 0$, hence P_3 is saddle point.
- The eigenvalues of P_4 are $\lambda_1 = -1$ $,\lambda_2 = k_2 > 0$ and $\lambda_3 = -\gamma < 0$ hence P_4 is also a saddle point.
- Next, the characteristic equation near P_5 is $\lambda^3 + A_5\lambda^2 + B_5\lambda + C_5 = 0$

Where

$$A_5 = (1-k_2)x_5 + b_1\frac{z_5}{(1+y_5)^2} + \left(\frac{h}{h+y_5-T}\right)^2 > 0 \text{ if } 1 > k_2$$

$$B_5 = -x_5\left[k_2 x_5 - b_1\frac{z_5}{(1+y_5)^2} - \left(\frac{h}{h+y_5-T}\right)^2\right] + k_1 k_2 x_5 y_5 + \frac{\gamma b_1 z_5}{(1+y_5)^2}$$

$$C_5 = \frac{\gamma b_1 y_5 z_5}{(1+y_5)^2} > 0.$$

By Routh-Hurwitz criterion this point is stable is $A_5 B_5 - C_5 > 0$

2.3. Piecewise Linear Threshold Policy Harvesting

In this section we take a piecewise linear threshold policy harvesting

$$H\,(y) = \begin{cases} 0 & \text{if} \quad y < T_1 \\ \dfrac{h\,(y - T_1)}{T_2 - T_1} & \text{if} \quad T_1 \le y \le T_2 \\ h & \text{if} \quad y > T_2 \end{cases} \qquad (5)$$

We shall study the effect of any small changes in the qualitative definition of this function.

The model in this case becomes

$$\begin{cases} \dot{x} = x\,(1-x) - k_1 xy \\ \dot{y} = k_2 xy - b_1\dfrac{yz}{1+y} - \dfrac{h\,(y - T_1)}{T_2 - T_1} \\ \dot{z} = z\left(-\gamma + b_2\dfrac{y}{1+y}\right) \end{cases} \qquad (6)$$

We do not consider the case when $y < T_1$ because we study that in (2.1) .

2.3.1. Equilibrium and Stability when $T_1 < y < T_2$

Theorem 3.1.1 The solution of the system (6) is bounded

Proof. Same proof in theorem 1.

The model system (6) has the following equilibrium points:

- The trivial equilibrium point $P_6(0,0,0)$ always exists.
- The equilibrium point $P_7 = (1,0,0)$ exists on the boundary of the first octant.
- The nontrivial equilibrium point $P_8(x_8, y_8, z_8)$, where

$$x_8 = 1 - k_1 y_8,\ y_8 = \frac{\gamma}{b_2 - \gamma},\ z_8 = \frac{b_2}{\gamma b_1}\left[k_2 x_8 y_8 - H\,(y)\right]$$

Exists under the conditions $1 > k_1 y_8$, $b_2 > \gamma$ and $k_2 x_8 y_8 > H\,(y)$.

The linear stability analysis about the above equilibrium points gives the following:

- The eigenvalues of P_6 are $\lambda_1 = 1, \lambda_2 = -\dfrac{h}{T_2 - T_1}$ and $\lambda_3 = -\gamma < 0$, therefore P_6 is saddle point.
- The eigenvalues of P_7 are $\lambda_1 = -1 < 0$ $,\lambda_2 = k_2 - \dfrac{h}{T_2 - T_1} > 0$ and $\lambda_3 = -\gamma < 0$, hence P_7 is stable point if $k_2 < \dfrac{h}{T_2 - T_1}$.
- Next, for the positive equilibrium P_8 , the variational matrix is

$$P_8 = \begin{bmatrix} -x_8 & -k_1 x_8 & 0 \\ k_2 y_8 & k_2 x_8 - b_1\dfrac{z_8}{(1+y_8)^2} - \dfrac{h}{T_2 - T_1} & -\dfrac{\gamma b_1}{b_2} \\ 0 & b_2\dfrac{z_8}{(1+y_8)^2} & 0 \end{bmatrix}$$

The characteristic equation at p_8 is $\lambda^3 + A_8\lambda^2 + B_8\lambda + C_8 = 0$

Where

$$A_8 = (1-k_2)x_8 + \frac{h}{T_2 - T_1} + b_1 \frac{z_8}{(1+y_8)^2},\ A_8 > 0 \text{ if } 1 > k_2$$

$$B_8 = -x_8(k_2 x_8 - b_1 \frac{z_8}{(1+y_8)^2} - \frac{h}{T_2 - T_1}) + k_1 k_2 x_8 y_8 + \frac{\gamma b_1 z_8}{(1+y_8)^2}$$

$$C_8 = \frac{\gamma b_1 x_8 z_8}{(1+y_8)^2} > 0$$

By Routh-Hurwitz criterion this point is stable is $A_8 B_8 - C_8 > 0$

2.3.2 Equilibrium and Stability when $y > T_2$

When the prey size is greater than the threshold value T_2, this means that $H(y) = h$ where h is a constant. In this case the model becomes

$$\begin{cases} x' = x(1-x) - k_1 xy \\ y' = k_2 xy - b_1 \dfrac{yz}{1+y} - h \\ z' = z(-\gamma + b_2 \dfrac{y}{1+y}) \end{cases} \quad (7)$$

In the same way we can prove the solution of system (7) is bounded and cannot have periodic orbit.

The model system (7) has the following equilibriums points:

- The trivial equilibrium point $P_9(0,0,0)$ always exists.
- The equilibrium point $P_{10}(1,0,0)$ exist on the boundary of the first octant.
- The nontrivial equilibrium point $P_{11}(x_{11}, y_{11}, z_{11})$.

  Where

$$x_{11} = 1 - k_1 y_{11},\ y_{11} = \frac{\gamma}{b_2 - \gamma},\ z_{11} = \frac{b_2}{\gamma b_1}[k_2 x_{11} y_{11} - h]\ .$$

Exists under the conditions $b_2 > \gamma$, $1 > k_1 y_{11}$ and $k_2 x_{11} y_{11} > h$

The linear stability analysis about the above equilibrium points gives the following:

- The eigenvalues of P_9 are $\lambda_1 = 1 > 0, \lambda_2 = 0$ and $\lambda_3 = -\gamma < 0$, hence P_9 is an unstable manifold along x -direction, while stable manifold along z -direction, Therefore P_9 is saddle point.
- The eigenvalues of P_{10} are $\lambda_1 = -1$, $\lambda_2 = k_2$ and $\lambda_3 = -\gamma$ hence P_{13} is also a saddle point.
- Next, the characteristic equation at p_{11} is

$$\lambda^3 + A_{11}\lambda^2 + B_{11}\lambda + C_{11} = 0$$

Where

$$A_{11} = (1-k_2)x_{11} + b_1 \frac{z_{11}}{(1+y_{11})^2},\ A_{11} > 0 \text{ if } 1 > k_2$$

$$B_{11} = k_1 k_2 x_{11} y_{11} + \frac{(\gamma + x_{11}) b_1 z_{11}}{(1+y_{11})^2} - k_2 x_{11}^2$$

$$C_{11} = \frac{\gamma b_1 x_{11} z_{11}}{(1+y_{11})^2} > 0$$

By Routh-Hurwitz criterion this point is stable is $A_{11} B_{11} - C_{11} > 0$

4. Invariant Region

We have studied the existence of equilibriums and determine the necessary and sufficient conditions for them; we also prove the bounds on the solution and show that these models cannot have any periodic solution in the interior of the positive quadrant of the xy -plane.

Next we would like to find the invariant region for these models.

By comparing the equilibrium points and their properties for both models in the cases where all populations can survive, we show the size of prey population is always is $1 - k_1 y_2$ also size of intermediate predator population is always $y = \frac{\gamma}{b_2 - \gamma}$, and the corresponding z values satisfy

$$z_2 \geq z_5 \geq z_8 \geq z_{11}$$

When

$$k_2 x_5 y_5 - H(y) \geq k_2 x_8 y_8 - H(y)$$

However $z_2 \geq z_5 \geq z_8 \geq z_{11}$

When

$$k_2 x_8 y_8 - H(y) \geq k_2 x_5 y_5 - H(y)$$

This difference depends on the distance between the size of prey population and the threshold, and whether the size of the top predator population increasing or decreasing depends on the harvest of prey. If we fixed all parameters, the size of intermediate predator and prey and are fixed. Therefore we can say the invariant region to these models is the intersection of the four cones where these cones are

$i.\ (x_2, y_2, z_2)$

$ii.\ (x_5, y_5, z_5)$

$iii.\ (x_8, y_8, z_8)$

$iv.\ (x_{11}, y_{11}, z_{11})$

Hence the invariant region is (x_{11}, y_{11}, z_{11}).

5. Numerical Simulation

In this section, we try to explain some of the findings of our study. After many numerical attempts and to guarantee all populations survive, we fixed the parameters as

$b_1 = 0.3391, b_2 = 0.1941, k_1 = 0.3475, k_2 = 0.1031, \gamma = 0.048$.

We found that the lowest value of the harvest is a good way to maintain the coexistence of all populations together and continued harvesting system tends to stability.

We take parameter of harvesting is 0.000001, 0.0209 and 0.364 and show in first and second values of harvesting all populations survive as in figures (1-2) and

in the third case we found that over-harvesting leads to extinction of intermediate predator and thus the extinction of top predator, as in figure (3).
And as a natural result the best policy for the harvest is controlled the harvest to ensure the survival of all populations and the sustainability of the harvest.
Also note that the harvest little means that the interactions between populations less than in the case of harvesting the biggest, and perhaps by the fact that the little harvest means enough food to all predator then no take long time to find the food, while when we increase harvest that's lead decreasing food then the predator need long time to find the food. This means the oscillations is small in first case figure (1) and big oscillations in second case figure (2).
At increased harvest the system tends to destabilize as in Figure (4), this proved to the role of harvesting in stability of this system.

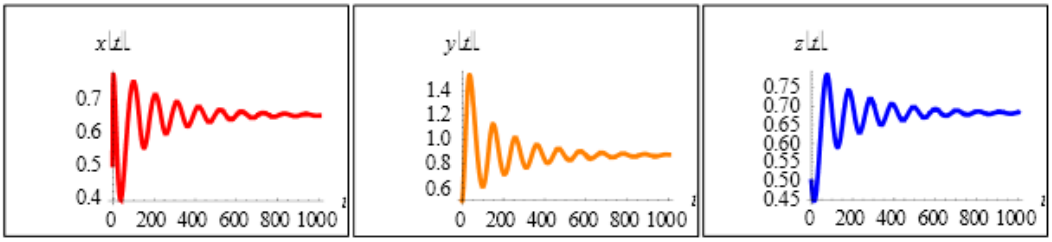

Figure 1. Oscillation of prey, intermediate and top predator for harvesting =0.000001

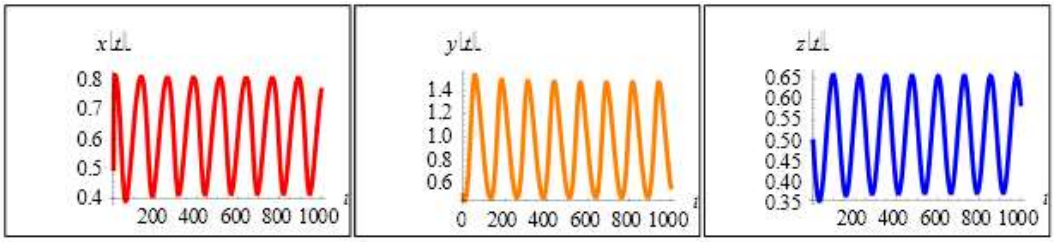

Figure 2. Oscillation of prey, intermediate and top predator for harvesting=0.0209

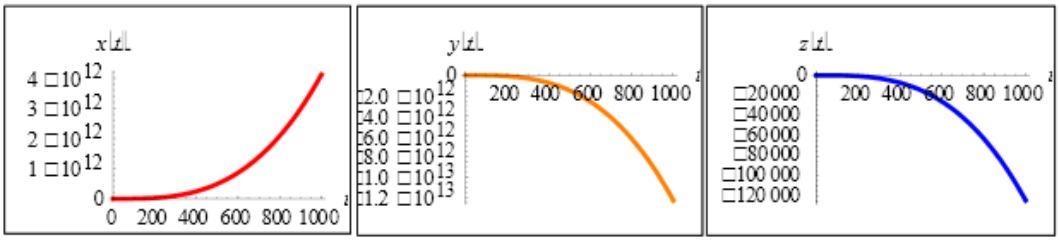

Figure 3. Oscillation of prey, intermediate and top predator for harvesting=0.364

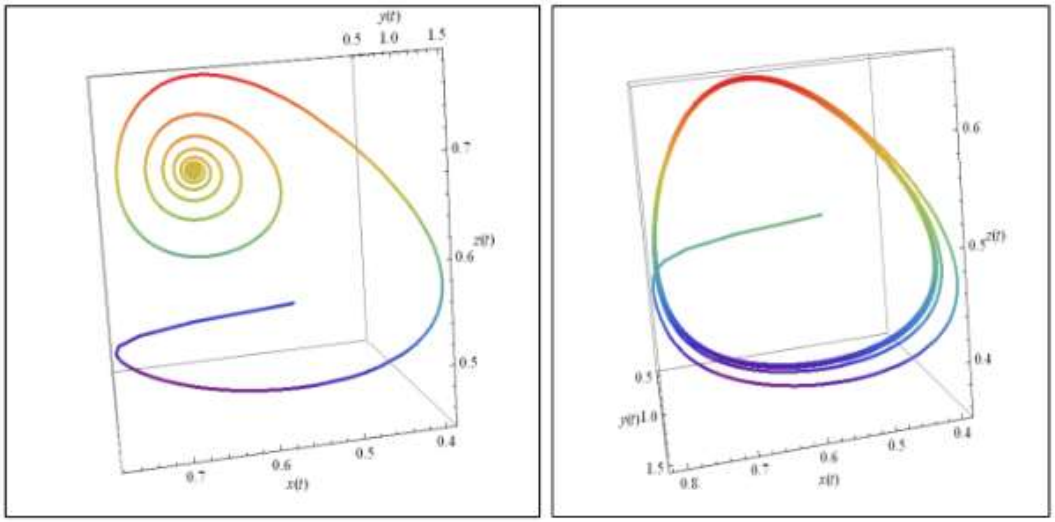

Figure 4. The trajectories of system is stability for parameter of harvesting 0.000001, and 0.029.

6. Discussion

We have studied two models with different continuous threshold harvesting functions. In the first model we consider a smooth harvesting behavior started when the size of intermediate predator reached the threshold, and in the second model we consider a

piecewise linear harvesting behavior after the size of intermediate predator reached the first threshold and continuous after that.

We show that the equilibriums and the dynamics of these are same except little different.

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العتبة المستمرة لحصاد المفترس المتوسط في نموذج سلسلة الأغذية

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الملخص

تقدم هذه الورقة نظام السلسلة الغذائية ثلاثة أنواع، والتي تتألف من عدد السكان فريسه و مفترس وسطي يعتمد في تغذيته على الفريسه بينما يعتمد المفترس على المترس الوسطي.

في هذا النموذج المفترس المتوسط يتعرض لخطر الحصاد, نحن نرس الحلول المقيدة ونقاط التوازن مع شروطها. كما تمت دراسة الاستقرار لكل نقطة من نقاط التوازن. علينا أن نحدد منطقة ثابتة، في هذه المنطقة نضمن عدم انقراض اي من المجتمعات الثلاثة مع استمرار الحصاد وضمان السيطرة على المرض وعدم تحوله الى وباء.