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## Efficient Solution of Third-Order ODEs with Diagonally Implicit RK Method Using Tri-Colored Tree and B-series Analysis

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### ABSTRACT

This study presents a pioneering approach to a new type of diagonally implicit Runge-Kutta-type (DIRKT) method developed specifically for solving third-order ODEs of this type.  $v''' = f(t, v, v')$ . These methods are called RKTG and are constructed using B series and Tri-colored tree theories to derive the required order conditions. We derive two methods: DRKTG4 and DRKTG5. The first achieves fourth-order accuracy in three stages, while the second reaches fifth order in four stages. Among other things, numerical experiments show these methods are accurate and efficient for solving these problems compared with existing Runge-Kutta methods, which posed these problems as first-order systems for testing.

**Keywords:** ordinary differential equations, numerical analysis, implicit Runge-Kutta, order criteria, B-series and tri-coloured.

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## الحل الفعال للمعادلة التفاضلية الاعتيادية من الدرجة الثالثة باستخدام طريقة رنج-كوتا الضمنية

### قطريًا باستخدام تحليل الشجرة ثلاثية الألوان والسلسلة B

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#### المخلص

تقدم هذه الدراسة نهجًا رائدًا لنوع جديد من طرق رنج كوتا نوع (DIRKT) الضمنية قطريًا والتي تم تطويرها خصيصًا لحل معادلات تفاضلية عادية من الدرجة الثالثة من النوع  $v''' = f(t, v, v')$ . تسمى هذه الطرق RKTG ويتم بناؤها باستخدام سلسلة B ونظريات الشجرة ثلاثية الألوان لاستنتاج شروط الدرجة المطلوبة. تم اشتقاق طريقتين: DRKTG4 و DRKTG5. يمكن تحقيق الطريقة الأولى بثلاث مراحل لدقتها من الدرجة الرابعة، بينما تصل الثانية إلى الدرجة الخامسة بأربع مراحل. من بين أمور أخرى، أثبتت هذه الطرق من خلال التجارب العددية أنها دقيقة وفعالة في حل المشكلات مقارنةً بطرق Runge-Kutta الحالية، والتي طرحت هذه الأنواع من المشكلات كأنظمة من الدرجة الأولى للاختبار

الكلمات المفتاحية / المعادلات التفاضلية العادية، التحليل العددي، رنج-كوتا الضمني، معايير الترتيب (الحالة)، السلسلة B والثلاثية الألوان.

## 1. INTRODUCTION

Third-order ordinary differential equations (ODEs) appear in several phenomena, such as electromagnetic waves, thin-film flow, gravity-driven flow, Propagation of acoustic waves in a relaxing medium, draining flows, and coating flows. <sup>(1-6)</sup> These physical models typically exhibit high-frequency oscillations, stiff behavior, or a need for high resolution at brief intervals. For example, electromagnetic wave propagation frequently results in stiff systems, particularly in dispersive media, necessitating high accuracy and numerical stability. Similarly, to adequately capture interface motion and form boundary layers, thin-film and coating processes typically require excellent space and time resolution. Gravity-driven or draining flows may incorporate nonlinearities and moving boundaries, making numerical management even more challenging; see <sup>(7, 8)</sup>.

This paper examines a numerical technique for solving a third-order initial value problem (IVP) for the given equation.

$$v''' = f(t, v, v') \quad (1)$$

with initial value problem

$$v(t_0) = v_0, \quad v'(t_0) = v'_0, \quad v''(t_0) = v''_0 \quad t \geq t_0$$

where the continuous vector-valued function  $f: R \times R^d \times R^d \rightarrow R^d$  does not contain derivatives of lower tr,  $v, v', v'' \in R^d$  and  $t \in R$ . Computing the general third order numerically necessitates more function evaluations, leading to heightened computational complexity and longer processing time. Here, we provide the implicit DIRKT formulas for solving third-order ordinary differential equations (ODEs) with the structure  $v$  prime prime prime equals f open paren t, v, v prime close paren. Numerical solutions are essential for solving third-order ordinary differential equations (ODEs). Researchers and scientists have used conventional techniques to evaluate higher-order ordinary differential equations (ODEs) by transforming them into a first-order format <sup>(9-11)</sup>.

Nevertheless, it would be more efficient to tackle these cases directly using numerical approaches. As a result, additional function evaluations must be computed, necessitating increased computational effort and longer processing time. Numerous academics have proposed direct numerical techniques to achieve more precise results in less computational time. However, experts in the field (12-17) have devised a multi-step approach that can solve third-order stiff differential equations. In contrast, (18-23) devised a technique using RK to solve special third-order ordinary differential equations (ODEs) directly.

The primary objective of this study is to use tri-colored trees theory to develop a one-step diagonally implicit RK methodology for solving third-order ODEs of this form.  $v''' = f(t, v, v')$ , which is denoted by RKTG.

This study addresses specific real-world problems, particularly electromagnetic waves, which are characterized by third-order ordinary differential equations (ODEs). We also show that the special method, RKTG, solves third-order ordinary differential equations (ODEs) more efficiently than the standard method, despite its greater complexity. The essay is structured as follows: Section 2 presents the formulation notion for the RKTG technique, which enables the direct integration of specified third-order ordinary differential equations (ODEs). In Section 3, we develop the novel theory of relevant colored trees and the corresponding B-series theory. Section 4 derives the order criteria for the DIRKT approach. Section 5 outlines the implementation of a three-stage DIRKT (Diagonally Implicit Runge-Kutta) technique for fourth-order methods and a four-stage approach for fifth-order methods. Section 6 presents numerical results demonstrating the superior efficiency and effectiveness of DIRKT approaches compared with methods currently used in scientific publications. Section 7 presents the final findings.

## 2. FORMULATION OF THE RKTG METHOD

The RKTG technique with  $l$ -stages for addressing the IVP (1) can be expressed in its general form as:

$$V_i = v_n + c_i h v'_n + \frac{1}{2} c_i^2 h^2 v''_n + h^3 \sum_{j=1}^l a_{ij} f(t_n + c_i h, V_j, V'_j) \quad 2$$

$$V'_i = v'_n + c_i h v''_n + h^2 \sum_{j=1}^l \bar{a}_{ij} f(t_n + c_j h, V_j, V'_j) \quad 3$$

$$v_{n+1} = v_n + h v'_n + \frac{1}{2} h^2 v''_n + h^3 \sum_{i=1}^l b_i f(t_n + c_i h, V_i, V'_i) \quad 4$$

$$v'_{n+1} = v'_n + h v''_n + h^2 \sum_{i=1}^l b'_i f(t_n + c_i h, V_i, V'_i) \quad 5$$

$$v''_{n+1} = v''_n + h \sum_{i=1}^l b''_i f(t_n + c_i h, V_i, V'_i) \quad 6$$

The RKTG method has parameters.  $a_{ij}$ ,  $\bar{a}_{ij}$ ,  $b_i$ ,  $b'_i$ ,  $b''_i$  and  $c_i$ . These parameters are defined for  $i, j = 1, 2, \dots, l$ . If the coefficient matrix of  $a_{ij}$  is lower triangular, the technique is considered explicit. Otherwise, it is implicit. The Kronecker product is a mathematical operation that is defined as follows:

$$V = e \otimes v_n + h(c \otimes v'_n) + \frac{h^2}{2}(c^2 \otimes v''_n) + h^3[A \otimes I_d]F(V, V') \quad 7$$

$$V' = e \otimes v'_n + h(c \otimes v''_n) + h^2[A \otimes I_d]F(V, V') \quad 8$$

$$v_1 = v_0 + h v'_0 + \frac{1}{2} h^2 v''_0 + h^3[b^T \otimes I_d]F(V, V') \quad 9$$

$$v'_1 = v'_0 + h v''_0 + h^2[b'^T \otimes I_d]F(V, V') \quad 10$$

$$v''_1 = v''_0 + h[b''^T \otimes I_d]F(V, V') \quad 11$$

$\exists \quad e = [1, \dots, 1]^T$  1-time,  $C = [c_m, m=1, \dots, l]^T$ ,  $b = [b_m, m=1, \dots, l]^T$ ,  $b' = [b'_m, m=1, \dots, l]^T$ ,  $b'' = [b''_m, m=1, \dots, l]^T$ ,  $A = [a_{ij}]^T$ ,  $\bar{A} = [\bar{a}_{ij}]^T$  denoted  $l \times l$  matrix while  $I_d$  denoted  $d \times d$  identity matrix. All parameters  $a_{ij}, \bar{a}_{ij}, c_i, b_i, b'_i, b''_i$ , are real numbers, and  $i, j = 1, 2, \dots, s$ . The RKTG techniques can be depicted using the Butcher tableau of scheme (2- 6); see (Table 1).

**Table 1: the Butcher tableau of RKTG techniques**

|          |          |          |                        |          |                |                |                        |                |
|----------|----------|----------|------------------------|----------|----------------|----------------|------------------------|----------------|
| $c_1$    | $a_{11}$ | $a_{12}$ | ...                    | $a_{1s}$ | $\bar{a}_{11}$ | $\bar{a}_{12}$ | ...                    | $\bar{a}_{1s}$ |
| $c_2$    | $a_{21}$ | $a_{22}$ | ...                    | $a_{2s}$ | $\bar{a}_{21}$ | $\bar{a}_{22}$ | ...                    | $\bar{a}_{2s}$ |
| $c_3$    | $a_{31}$ | $a_{32}$ | ...                    | $a_{3s}$ | $\bar{a}_{31}$ | $\bar{a}_{32}$ | ...                    | $\bar{a}_{3s}$ |
| $\vdots$ |          |          | $\vdots \vdots \vdots$ |          |                |                | $\vdots \vdots \vdots$ |                |
| $c_s$    | $a_{s1}$ | $a_{s2}$ | ...                    | $a_{ss}$ | $\bar{a}_{s1}$ | $\bar{a}_{s2}$ | ...                    | $\bar{a}_{ss}$ |
|          | $b_1$    | $b_2$    | ...                    | $b_s$    |                |                |                        |                |
|          | $b'_1$   | $b'_2$   | ...                    | $b'_s$   |                |                |                        |                |
|          | $b''_1$  | $b''_2$  | ...                    | $b''_s$  |                |                |                        |                |

### 3. THE ORDER CONDITIONS BY TRI-COLORED TREES AND B-SERIES FOR RKTG TECHNIQUE

This section will mention fundamental definitions, lemmas, and related theorems. This article relies

**Definition3.1:** The RKTG technique has an order of  $q$  when used for a third-order ordinary differential equation (ODE) (1) with the assumption that  $v(t_n)=v_n$ ,  $v'(t_n)=v'_n$ ,  $v''(t_n)=v''_n$ . Hence, the local truncation error norms of the exact solution and its derivatives up to order three must satisfy the given criteria.

$$v(t_n + h) - v_{n+1} = O(h^{q+1})$$

$$v'(t_n + h) - v'_{n+1} = O(h^{q+1})$$

$$v''(t_n + h) - v''_{n+1} = O(h^{q+1}) \quad 12$$

To formulate algebraic order conditions for the RKTG technique, we utilize the independent form of a three-order IVP.

$$v'''(t) = f(v(t), v'(t)) \quad 13$$

with initial conditions

$$v(t_n) = v_n, \quad v'(t_n) = v'_n, \quad v''(t_n) = v''_n$$

The third-order IVP (1) The presentation of the problem can be extended by incorporating  $t=z$ , which has a one-dimensional structure, into the IVP (1):

$$z''' = 0 \quad 14$$

$$v''' = f(t, v, v') \quad 15$$

$$z(t_n) = z_n = t_n, \quad z'(t_n) = z'_n = 1, \quad z''(t_n) = z''_n = 0 \quad 16$$

$$v(t_n) = v_n = t_n, \quad v'(t_n) = v'_n, \quad v''(t_n) = v''_n \quad 17$$

using RKTG method (2)-(6) to the structure (14)-(17) we get

$$Z_i = z_n + c_i h z'_n + \frac{1}{2} c_i^2 h^2 z''_n \quad 18$$

$$V_i = v_n + c_i h v'_n + \frac{1}{2} c_i^2 h^2 v''_n + h^3 \sum_{j=1}^l \bar{a}_{ij} f(Z_j, V_j, V'_j) \quad 19$$

$$z_{n+1} = z_n + c_i h z'_n + \frac{1}{2} c_i^2 h^2 z''_n \quad 20$$

$$z'_{n+1} = z'_n + c_i h z''_n \quad 21$$

$$z''_{n+1} = z''_n \quad 22$$

$$v_i = v_n + c_i h v'_n + \frac{1}{2} c_i^2 h^2 v''_n + h^3 \sum_{i=1}^l b_i f(Z_i, V_i, V'_i) \quad 23$$

$$v'_i = v'_n + c_i h v''_n + h^2 \sum_{i=1}^l b'_i f(Z_i, V_i, V'_i) \quad 24$$

$$v''_i = v''_n + h \sum_{i=1}^l b''_i f(Z_i, V_i, V'_i) \quad 25$$

By substituting equation (16) into the system of equations (18)–(25), we obtain

$$Z_i = t_n + c_i h \quad 26$$

$$z_{n+1} = t_n + h \quad 27$$

$$z'_{n+1} = 1 \quad 28$$

$$z''_{n+1} = 0 \quad 29$$

$$v_{n+1} = v_n + h v'_n + \frac{1}{2} h^2 v''_n + h^3 \sum_{i=1}^l b_i f(t_n + c_i h, V_i, V'_i) \quad 30$$

$$v'_{n+1} = v'_n + h v''_n + h^2 \sum_{i=1}^l b'_i f(t_n + c_i h, V_i, V'_i) \quad 31$$

$$v''_{n+1} = v''_n + h \sum_{i=1}^l b''_i f(t_n + c_i h, V_i, V'_i) \quad 32$$

$$V_i = v_n + c_i h v'_n + \frac{1}{2} c_i^2 h^2 v''_n + h^3 \sum_{i,j=1}^l \bar{a}_{ij} f(t_n + c_i h, V_j, V'_j) \quad 33$$

It may be inferred that equations (30)–(33) are equivalent to equations (2)–(6). The RKTG technique is employed to resolve the interdependent equations. Then, it suffices to examine the numerical solutions of the independent form (13); thus, the RKTG approach (2)–(6) can be assumed in the following form.

$$v_{n+1} = v_n + h v'_n + \frac{1}{2} h^2 v''_n + h^3 \sum_{i=1}^l b_i f(V_i, V'_i) \quad 34$$

$$v'_{n+1} = v'_n + h v''_n + h^2 \sum_{i=1}^l b'_i f(V_i, V'_i) \quad 35$$

$$v''_{n+1} = v''_n + h \sum_{i=1}^l b''_i f(V_i, V'_i) \quad 36$$

$$V_i = v_n + c_i h v'_n + \frac{1}{2} c_i^2 h^2 v''_n + h^3 \sum_{i,j=1}^l \bar{a}_{ij} f(V_j, V'_j) \quad 37$$

Using the analytic solution  $v(t)$  and the elementary differential notation, the following elementary differentials are obtained:

$$v^{(1)} = v', \quad v^{(2)} = v'', \quad v^{(3)} = f$$

$$v^{(4)} = f_{vv'} v' + f_{v'} v''$$

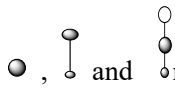
$$v^{(5)} = f_{vv}(v')^2 + 2f_{vv'} v' v'' + f_{v'v'} (v'')^2 + f_{v'} v'' + f_{v'} f$$

$$v^{(6)} = 3f_{vv} v' v'' + f_{vvv} (v')^3 + 3f_{vvv'} v' (v')^2 + 3f_{vv'} (v'')^2 + 3f_{vv'} v' f + 3f_{vv'} v' (v'')^2 + 3f_{v'v'} v'' f + f_{v'v'} (v'')^3 + f_{v'} f_{v'} v' + f_{v'} f_{v'} v'' + f_{v'} f$$

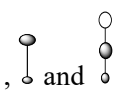
$$v^{(7)} = 3f_{vv}(v'')^2 + 4f_{vv}v'f + 6f_{vvv}(v')^2v'' + 12f_{vvv}(v'')^2f + f_{vvvv}(v')^4 + 4f_{vvvv}v''(v')^3 + 6f_{vvv}(v')^2f + 10f_{vv}v''f + 6f_{vvv}(v'')^3 + 4f_{vv}f_v(v')^2 + 6f_{vv}f_vv'v'' + 12f_{vvv}v'v''f + 6f_{vvv}(v'')^2(v')^2 + 4f_{vvv}(v'')^3v' + 3f_{vv}f^2 + 4f_{vv}f_vv'v'' + 5f_{vv}f_v(v'')^2 + 6f_{vv}(v'')^2f + f_{vvv}(v'')^4 + 2f_{vv}f_vv'' + f_{vv}f_v(v')^2 + f_{vv}f_vf + f_vf_vv' \quad 38$$

we omitted the parameters  $(t_0)$ ,  $v(t_0)$  on two side left and right respectively. We notice that the higher the rank, the more complex it becomes. Based by (24) To address differences in the order of differential equations, we propose a geometric form. The theorem is referred to as the tri-colored trees theorem, modified specifically for third-order differential equations. Three sorts of vertices were utilized: "tiny gray", "large gray", and "large white" vertices, which are interconnected by branches. where the large gray represents  $v'$  and the tiny gray represents the end of  $v'$ ; at last, the large white vertex represents  $f$  the last one, "white vertex," with 1 branches going upwards, represents the  $l$ th derivative of  $f$  about  $v$ .

**Definition3.2:** The group of tri-colored trees ( $\mathbb{T}^T$ ) is frequently described in the following manner:

The graphs  represented by  $\mathbb{T}^T$ . If  $t_1, \dots, t_l$  are elements of set  $\mathbb{T}^T$ , then the graph formed by linking the roots of  $t_1, \dots, t_l$  to the larger white

vertex of the graph  is also in group  $\mathbb{T}^T$ . The tree is indicated by  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$  The tiny gray vertex at the bottom serves as the root of the structure. The abbreviation "3" indicates that the roots  $t_1, \dots, t_l$  are trees that compose a graph with three vertices.

We marked the trees  by  $Y_1$ ,  $Y_2$  and  $Y_3$  respectively. The guidelines for producing tri-colored trees are listed below:

The permanent root is tiny gray

A tiny grey vertex has a single offspring, which must be large gray

A larger grey vertex has a single offspring, which must be a large white entity.

All offspring of a large white vertex need to be tiny gray.

**Definition3.3:** the function  $g : \mathbb{T}^T \rightarrow \mathbb{N}$  is an integer-valued, and it is frequently designated as:

$$g(Y_1) = 1; \quad , \quad g(Y_2) = 2; \quad , \quad g(Y_3) = 3;$$

$$\text{For } t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T \quad g(t) = \left[ \sum_{i=1}^r g(t_i) + \sum_{i=r+1}^l (g(t_i) - 1) \right] + 3.$$

For every element  $t$  in  $\mathbb{T}^T$ , the order of  $t$  is denoted by. The function  $g(t)$  denotes the total number of nodes within  $t$ . A group containing all elements with an order of  $q$  is represented as  $\mathbb{T}^T_q$ .

**Definition3.4:**  $\forall t \in \mathbb{T}^T$ , the primary differential of the function  $F(t) : R^d \times R^d \times R^d \rightarrow R^d$  is a vector-valued function, which is frequently defined as

$$F(Y_1)(v, v', v'') = v' \quad , \quad F(Y_2)(v, v', v'') = v'' \quad , \quad F(Y_3)(v, v', v'') = f(v)$$

$$\text{For } t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T$$

$$F(t)(v) = f^l(v)(F(t_1)(v, v', v''), \dots, F(t_l)(v, v', v''))$$

**Definition3.5:** The symmetry factors  $\phi : \mathbb{T}^T \rightarrow \mathbb{N}$  are frequently defined as:

$$\phi(Y_1) = 1 \quad , \quad \phi(Y_2) = 1 \quad , \quad \phi(Y_3) = 1$$

$$\text{For } t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T \quad \phi(t) = (\mu_1! \mu_2! \dots) \prod_{i=1}^l \phi(t_i)$$

Where  $\mu_1, \mu_2, \dots$  count similar trees within  $t_1, \dots, t_l$ . via the B-series, we may represent rooted trees in the context of RK techniques given in (25), may deduce the B-series for the set of 3-colored trees used in RKTG methods.

**Definition3.6:** For a function  $\varphi : \mathbb{T}^T \cup \{\emptyset\} \rightarrow R$  a formal series of the model

$$B(\varphi, v, v') = \varphi(\emptyset)v +$$

$$\sum_{t \in \mathbb{T}^T} \frac{h^{g(t)}}{\phi(t)} \varphi(t) F(t)(v, v', v'') \quad \text{is named a B-series.}$$

The fundamental lemma is essential for constructing the order criteria for RKTG. techniques.

**fulfills.1:** A B-series  $B(\varphi, v, v')$  with factors mapping  $\varphi$  that fulfils the given condition  $\varphi(\emptyset) = 1$ , Then

$$\left[ h^3 f \left( B(\varphi, v, v'), B \left( \frac{g}{h} \varphi', v, v' \right) \right) \right] \quad \text{is Bseries again} \left[ h^3 f(B(\varphi, v, v'), B \left( \frac{g}{h} \varphi', v, v' \right)) = \right]$$

$B(\varphi''', v, v')$  when  $\varphi'''(\emptyset) = \varphi'''(Y_1) = \varphi'''(Y_2) = 0$ , but  $\varphi'''(Y_3) = 1$  and for each  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$  with  $g \geq 4$   $\varphi'''(t) = \prod_{i=1}^r \varphi(t_i) \prod_{r+1}^l \varphi'(t_i) g(t_i)$ .

**Proof:** since  $B(\varphi, v, v') = v + O(\hbar)$  and  $B\left(\frac{g}{\hbar} \varphi', v, v'\right) = v' + O(\hbar)$  then the Taylor expansion  $\hbar^3 f(B(\varphi, v, v'), B(\frac{g}{\hbar} \varphi', v, v'))$  show

that  $\hbar^3 f\left(B(\varphi, v, v'), B\left(\frac{g}{\hbar} \varphi', v, v'\right)\right) = \hbar^3 f(v, v') + O(\hbar^4)$  which implies that  $\varphi'''(\emptyset) = \varphi'''(Y_1) = \varphi'''(Y_2) = 0$ , but  $\varphi'''(Y_3) = 1$

Proceeding as in the proof<sup>(19)</sup>, we have :

$$\hbar^3 f\left(B(\varphi, v, v'), B\left(\frac{g}{\hbar} \varphi', v, v'\right)\right) = \hbar^3 \sum_{l \geq 0} \sum_{r=0}^l \frac{l!}{r!(l-r)!} \frac{\partial^l f}{\partial v^r \partial v'^{l-r}}(v, v')(B(\varphi, v, v') - v)^r \left(B\left(\frac{g}{\hbar} \varphi', v, v'\right) - v'\right)^{l-r}$$

$$\sum_{l \geq 0} \sum_{r=0}^l \frac{l!}{r!(l-r)!} \sum_{t_1 \in \mathbb{T}^r} \dots \sum_{t_r \in \mathbb{T}^r} \sum_{t_{r+1} \in \mathbb{T}^{l-r}/\{Y_1, Y_2\}} \dots \sum_{t_l \in \mathbb{T}^{l-r}/\{Y_1, Y_2\}} \frac{\hbar^{g(t_1)+g(t_2)+\dots+g(t_l)-(l-r)}}{\phi(t_1) \dots \phi(t_l)} e(t) F(v(t_1, v', v''), \dots, v(t_l, v', v'')) g(t_l)$$

$$\varphi(t_1) \dots \varphi(t_r) \varphi'(t_{r+1}) \dots \varphi'(t_l) \frac{\partial^l f}{\partial v^r \partial v'^{l-r}}(v, v') (F(t_1)(v, v', v'') \dots F(t_l)(v, v', v''))$$

$$= \sum_{l \geq 0} \sum_{r=0}^l \sum_{t_1 \in \mathbb{T}^r} \dots \sum_{t_r \in \mathbb{T}^r} \sum_{t_{r+1} \in \mathbb{T}^{l-r}/\{Y_1, Y_2\}} \dots \sum_{t_l \in \mathbb{T}^{l-r}/\{Y_1, Y_2\}} \frac{\hbar^{g(t)} l! \mu_1! \mu_2! \dots \xi_1! \xi_2! \dots}{\phi(t) r!(l-r)!} g(t_{r+1}) \dots g(t_l) \quad 39$$

$$\varphi(t_1) \dots \varphi(t_r) \varphi'(t_{r+1}) \dots \varphi'(t_l) F(t)(v, v', v'')$$

$$= \sum_{t \in \mathbb{T}^r, g(t) \geq 4} \frac{\hbar^{g(t)}}{\phi(t)} g(t_{r+1}) \dots g(t_l) \varphi(t_1) \dots \varphi(t_r) \varphi'(t_{r+1}) \dots \varphi'(t_l) F(t)(v, v', v'')$$

where, unique equality  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$  and the approaches for ranking the subtrees  $t_1 \dots t_l$  in  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$ , the multiplicity of  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$  is  $\frac{r!(l-r)!}{l! \mu_1! \mu_2! \dots \xi_1! \xi_2! \dots}$  where  $\mu_1, \mu_2 \dots$  the number of trees among  $t_1, \dots, t_r$  and  $\xi_1, \xi_2 \dots$  the number of trees among  $t_{r+1}, \dots, t_l$  then we have

$$\varphi'''(t) = \prod_{i=1}^r \varphi(t_i) \prod_{r+1}^l \varphi'(t_i) g(t_i)$$

We get

$$\begin{aligned} \hbar^3 f(B(\varphi, v, v'), B(\frac{g}{\hbar} \varphi', v, v')) &= \sum_{t \in \mathbb{T}^r} \frac{\hbar^{g(t)}}{\phi(t)} \varphi''' F(t)(v, v', v'') \\ &= B(\varphi''', v, v') \end{aligned}$$

**Theorem 3.1:** Let's consider the analytic solution  $v(t_0 + \hbar)$  of (13) is a B-series  $B(e, v_0, v'_0)$  since  $e$  is a real function defined on  $\mathbb{T}^r \cup \{\emptyset\}$  then

$$\begin{aligned} e(\emptyset) &= 1, \quad e(Y_1) = 1, \quad e(Y_2) = \frac{1}{2}, \quad e(Y_3) \\ &= \frac{1}{6} \end{aligned}$$

And for  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3$

$$\begin{aligned} e(t) &= \frac{1}{g(t)(g(t)-1)(g(t)-2)} \prod_{i=1}^r e(t_i) \prod_{r+1}^l e(t_i) g(t_i) \end{aligned}$$

Proof:

$$\begin{aligned} v(t_0 + \hbar) &= B(e, v_0, v'_0) = e(\emptyset)v_0 + \hbar e(Y_1)v'_0 + \\ &\hbar^2 e(Y_2)v''_0 + \hbar^3 e(Y_3)v'''_0 + \end{aligned}$$

$$\sum_{t_{r+1} \in \mathbb{T}^r/\{Y_1, Y_2, Y_3\}} \frac{\hbar^{g(t)} \hbar^{g(t)-1}}{\phi(t)} e(t) F(v_0, v'_0, v''_0) =$$

$$B\left(\frac{g}{\hbar}, v_0, v'_0\right)$$

Therefore, we differentiate.  $v(t_0 + \hbar)$  three times, and it is represented by

$$\begin{aligned} \frac{d^3}{d\hbar^3} [v(t_0 + \hbar)] &= e(Y_1)v'_0 + \\ &2\hbar e(Y_2)v''_0 + 3\hbar^2 e(Y_3)v'''_0 + \\ &\sum_{t_{r+1} \in \mathbb{T}^r/\{Y_1, Y_2, Y_3\}} \frac{g(t) \hbar^{g(t)-1}}{\phi(t)} e(t) F(v_0, v'_0, v''_0) = \end{aligned}$$

$$\begin{aligned} v(t_0 + \hbar)'' &= \frac{d^2}{d\hbar^2} [v(t_0 + \hbar)] = 2e(Y_2)v''_0 + \\ &6\hbar e(Y_3)v'''_0 + \\ &\sum_{t_{r+1} \in \mathbb{T}^r/\{Y_1, Y_2, Y_3\}} \frac{g(t)(g(t)-1)\hbar^{g(t)-2}}{\phi(t)} e(t) F(v_0, v'_0, v''_0) \end{aligned}$$

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$$v(t_0 + h)''' = \frac{d^3}{dh^3} [v(t_0 + h)] = 6e(Y_3)v_0''' + \sum_{t_{r+1} \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}} \frac{g(t)(g(t)-1)(g(t)-2)h^{g(t)-3}}{\phi(t)} e(t)F(v_0, v_0', v_0'')$$

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Moreover, from Lemma 3.1 we get

$$f(B(e, v_0, v_0'), B\left(\frac{g}{h}\bar{\varphi}, v_0, v_0'\right) = e^{(3)}(Y_3)f(v_0, v_0') + \sum_{t_{r+1} \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}} \frac{h^{g(t)-3}}{\phi(t)} e^{(3)}(t)F(v_0, v_0', v_0'')$$

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Where,  $e^{(3)}(Y_3) = 1$  and  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}$

$$e^{(3)}(Y_3) = \prod_{i=1}^r e(t_i) \prod_{r+1}^l e(t_i)g(t_i)$$

Substituting (42), (43) into equation (13) on both sides, by comparing the parameters of the identical elementary differential, we obtain:

$$e^{(3)}(Y_3) = \frac{1}{6}$$

And  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}$

$e(t)$

$$= \frac{1}{g(t)(g(t)-1)(g(t)-2)} \prod_{i=1}^r e(t_i) \prod_{r+1}^l e(t_i)g(t_i)$$

At the end, and by Taylor series expansions of  $v(t_0 + h)$  where  $h=0$  and  $(\emptyset) = e(Y_1) = 1$ ,  $e(Y_2) = \frac{1}{2}$ ,  $e(Y_3) = \frac{1}{6}$ .

$\forall t \in \mathbb{T}^T$  We can express the density as  $\gamma(t) = \frac{1}{e(t)}$

and write  $\{Z^+\}$  as following  $\alpha(t) = \frac{g(t)!}{\phi(t)\gamma(t)}$  then by the (theorem3.1) we get propositions [3.1.1 and 3.1.2] as following:

**Proposition 3.1.1:** for all  $t \in \mathbb{T}^T$ , the density  $\gamma(t)$  is a positive integer-valued-function on  $\mathbb{T}^T$  satisfying

$$\gamma(Y_1) = 1, \quad \gamma(Y_2) = 2, \quad \gamma(Y_3) = 6$$

$$t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T$$

$$\gamma(t) = g(t)(g(t)-1)(g(t)-2)$$

$$- 2) \prod_{i=1}^r \gamma(t_i) \prod_{r+1}^l \frac{\gamma(t_i)}{g(t_i)}$$

**Proposition3.1.2:** for all  $t \in \mathbb{T}^T$   $\alpha(t)$  is a non-negative integer satisfying

$$\alpha(Y_1) = 1, \quad \alpha(Y_2) = 1, \quad \alpha(Y_3) = 1$$

$t = \{t_1^{\mu_1}, \dots, t_r^{\mu_r} < t_{r+1}^{\mu_{r+1}}, \dots, t_l^{\mu_{l+1}} > \}_3 \in \mathbb{T}^T$  with  $t_1, \dots, t_r$  distinct  $t_{r+1}, \dots, t_l$  distinct

$\alpha(t)$

$$= (g(t))$$

$$- 3)! \prod_{i=1}^r \frac{1}{\mu_i!} \left(\frac{\alpha(t_i)}{g(t_i)!}\right)^{\mu_i} \prod_{i=r+1}^l \frac{1}{\mu_i!} \left(\frac{\alpha(t_i)}{(g(t_i)-1)!}\right)^{\mu_i}$$

The symbol  $\mu_i$  denotes the number of times  $t_i, \exists i=1, \dots, l$

Thus, we can write the B-series in Definition3.6 as follows:

$$B(\varphi, v, v') = \varphi(\emptyset)v +$$

$$\sum_{t \in \mathbb{T}^T} \frac{h^{g(t)-3}}{\phi(t)} \varphi(t) \alpha(t) \gamma(t) F(t)(v, v', v'')$$

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and  $f(B(\varphi, v, v'), B\left(\frac{g}{h}\bar{\varphi}, v, v'\right))$  can be expressed as

$$f(B(\varphi, v, v'), B\left(\frac{g}{h}\bar{\varphi}, v, v'\right)) =$$

$$\sum_{t \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}} \frac{h^{g(t)-3}}{g(t)!} \varphi^{(3)}(t) \alpha(t) \gamma(t) F(t)(v, v', v'')$$

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**Theorem3.2:** The analytic solution and derivative  $v(t_0 + h)$ ,  $v'(t_0 + h)$  and  $v''(t_0 + h)$  of equation (13) have B-series as follows:

$$v(t_0 + h) = v_0 + \sum_{t_{r+1} \in \mathbb{T}^T} \frac{h^{g(t)}}{g(t)!} \alpha(t) F(v_0, v_0', v_0'')$$

$$= B\left(\frac{\alpha(t)\phi}{g!}, v_0, v_0'\right)$$

$$= B\left(\frac{1}{\gamma(t)}, v_0, v_0'\right)$$

$$\begin{aligned}
 v'(t_0 + h) &= \sum_{t_{r+1} \in \mathbb{T}^T} \frac{h^{g(t)-1}}{(g(t)-1)!} \alpha(t) F(v_0, v'_0, v''_0) \\
 &= B\left(\frac{\alpha(t)\phi}{h(g-1)!}, v_0, v'_0\right) \\
 &= B\left(\frac{g}{h\gamma(t)}, v_0, v'_0\right) \\
 v''(t_0 + h) &= \sum_{t_{r+1} \in \mathbb{T}^T} \frac{h^{g(t)-2}}{(g(t)-2)!} \alpha(t) F(v_0, v'_0, v''_0) \\
 &= B\left(\frac{\alpha(t)\phi}{h^2(g-2)!}, v_0, v'_0\right) \\
 &= B\left(\frac{g(g-1)}{h^2\gamma(t)}, v_0, v'_0\right)
 \end{aligned}$$

Now, to form the numerical resolution and derivatives.  $v_1 v'_1, v''_1$  of equation (13) engender the RKTG approach (2)-(6); suppose that  $V_i$  and  $V'_i$  in (2),(3), the widening of the B series can be represented as  $V_i = B(\vartheta_i, v_0, v'_0)$  and  $V'_i = B\left(\frac{g}{h}\bar{\vartheta}_i, v_0, v'_0\right)$  Respectively, the first two formulas enclosed within brackets (2)-(6) have been transformed as follows:

$$\begin{aligned}
 &B(\vartheta_i, v_0, v'_0) \\
 &= v_0 + c_i h v'_0 + \frac{1}{2} c_i^2 h^2 v''_0 \\
 &+ h^3 \sum_{j=1}^l a_{ij} f\left(B(\vartheta_i, v_0, v'_0), B\left(\frac{g}{h}\bar{\vartheta}_i, v_0, v'_0\right)\right) \\
 &B\left(\frac{g}{h}\bar{\vartheta}_i, v_0, v'_0\right) \\
 &= v'_0 + c_i h v''_0 \\
 &+ h^2 \sum_{j=1}^l \bar{a}_{ij} f\left(B(\vartheta_i, v_0, v'_0), B\left(\frac{g}{h}\bar{\vartheta}_i, v_0, v'_0\right)\right)
 \end{aligned}$$

By (44), (45), the former two equations can be written as follows:

$$\begin{aligned}
 \vartheta_i(\mathfrak{z})v_0 + \sum_{t \in \mathbb{T}^T} \frac{h^{g(t)}}{g(t)!} \vartheta_i(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\
 = v_0 + c_i h v'_0 + \frac{1}{2} c_i^2 h^2 v''_0 \\
 + h^3 \sum_{j=1}^l \sum_{t \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}} \frac{h^{g(t)}}{g(t)!} a_{ij} \vartheta_j^{(3)} \alpha(t) \gamma(t) F(t)(v, v', v'')
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{t \in \mathbb{T}^T} \frac{h^{g(t)-1}}{(g(t)-1)!} \bar{\vartheta}_i(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\
 &= v'_0 + c_i h v''_0 \\
 &+ h^2 \sum_{j=1}^l \sum_{t \in \mathbb{T}^T / \{Y_1 Y_2 Y_3\}} \frac{h^{g(t)-1}}{g(t)!} \bar{a}_{ij} \vartheta_i^{(3)} \alpha(t) \gamma(t) F(t)(v, v', v'')
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \vartheta_i(\emptyset) &= 1, \quad \vartheta_i(Y_1) = c_i, \quad \vartheta_i(Y_2) = \frac{1}{2} c_i^2 \\
 \vartheta_i(Y_3) &= \sum a_{ij} \vartheta_i'''(Y_3), \quad \vartheta_i(Y_3) = \sum_{j=1}^l a_{ij}
 \end{aligned}$$

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$$\begin{aligned}
 \bar{\vartheta}_i(Y_1) &= 1, \quad \bar{\vartheta}_i(Y_2) = c_i, \quad \bar{\vartheta}_i(Y_3) = \\
 &\frac{1}{3} \sum_{j=1}^l \bar{a}_{ij} \vartheta_j'''(Y_3) = \frac{1}{3} \sum \bar{a}_{ij}
 \end{aligned}$$

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And

$$\begin{aligned}
 \vartheta_i(t) &= \sum_{j=1}^l a_{ij} \vartheta_j'''(t), \quad \bar{\vartheta}_i(t) = \\
 &\sum_{j=1}^l \frac{\bar{a}_{ij}}{g(t)} \vartheta_j'''(t)
 \end{aligned}$$

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In addition, for trees  $t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \}_3 \in \mathbb{T}^T$  and  $g(t) \geq 4$  by Lemma 3.1, given

$$\vartheta_j'''(t) = \prod_{i=1}^r \vartheta_j(t_i) \prod_{i=r+1}^l \bar{\vartheta}_j(t_i)$$

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by compensation (48) into (49), we get

$$\begin{aligned}
 \vartheta_j'''(t) &= \\
 &\prod_{i=1}^r [\sum_{j=1}^l a_{ij} \vartheta_j'''(t_i)] \prod_{i=r+1}^l \left[ \sum_{j=1}^l \frac{\bar{a}_{ij}}{g(t)} \vartheta_j'''(t_i) \right]
 \end{aligned}$$

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Denote,  $\vartheta_j'''(t) = \omega_j(t) \quad \forall t = \{t_1, \dots, t_r < t_{r+1}, \dots, t_l > \} \in \mathbb{T}\mathbb{T}$  and  $(t) \geq 4$ . Then (50) can be expressed as follows.

$$\omega_j(t) = \prod_{i=1}^r \left[ \sum_{j=1}^l a_{ij} \omega_j(t_i) \right] \prod_{i=r+1}^l \left[ \sum_{j=1}^l \frac{\bar{a}_{ij}}{g(t)} \omega_j(t_i) \right]$$

In general, the next lemma is important to give the values of  $\omega_j(Y) \quad \forall t \in \mathbb{T}\mathbb{T}/\{Y_1 Y_2 Y_3\}$

**Lemma 3.2:** The function  $\omega_j(t)$  on  $\mathbb{T}\mathbb{T}/\{Y_1, Y_2, Y_3\}$  can be computed recursively as

$$\omega_j(Y_3) = 1$$

For  $t = \{t_1^{\mu_1}, \dots, t_r^{\mu_r} < t_{r+1}^{\mu_{r+1}}, \dots, t_l^{\mu_{l+1}} > \} \in \mathbb{T}\mathbb{T}$  with  $t_1, \dots, t_l$  distinct and different from  $Y_1, Y_2$

$$\omega_j(t) = \frac{1}{2^{\mu_2}} c_i^{\mu_1+2\mu_2} \prod_{k=3}^r \left[ \sum_{j=1}^l a_{ik} \omega_j(t_k) \right]^{\mu_k} \prod_{i=r+1}^l \left[ \sum_{j=1}^l \frac{\bar{a}_{ij}}{g(t)} \omega_j(t_i) \right]^{\mu_k}$$

we denote  $\omega(t) = (\omega_1(t), \dots, \omega_l(t))^T$ , for  $t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}$ , the preliminary weights associated with  $v_1, v'_1$  and  $v''_1$  respectively defined by:

$$\theta(t) = \sum_{i=1}^l b_i \omega_i(t) = b^T \omega(t)$$

$$\theta'(t) = \sum_{i=1}^l b'_i \omega_i(t) = b'^T \omega(t)$$

$$\theta''(t) = \sum_{i=1}^l b''_i \omega_i(t) = b''^T \omega(t)$$

**Theorem 3.3:** The numerical resolution and derivatives  $v_1, v'_1, v''_1$  of equation (13) produced by the RKTG method (2)-(6) have the following B-series

$$\begin{aligned} v_1(t_0 + h) &= v_0 + h v'_0 + \frac{1}{2} h^2 v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)}}{g(t)!} \theta(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\ v'_1(t_0 + h) &= v'_0 + h v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)-1}}{g(t)!} \theta'(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\ v''_1(t_0 + h) &= v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)-2}}{g(t)!} \theta''(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \end{aligned}$$

Proof: Suppose  $V_i$  and  $V'_i$  in equations (2)-(6) are B-series

$$B(\vartheta_i, v_0, v'_0), B(\frac{g}{h} \vartheta_i, v_0, v'_0)$$

respectively. By Lemma (3.1), we have

$$\begin{aligned} h^3 F(V_i, V'_i) &= B(\vartheta_i''', v_0, v'_0) \\ &= \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)}}{g(t)!} \vartheta_i'''(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \end{aligned}$$

Therefore

$$\begin{aligned} v_1(t_0 + h) &= v_0 + h v'_0 + \frac{1}{2} h^2 v''_0 + \sum_{i=1}^l b_i B(\vartheta_i''', v_0, v'_0) \\ &= v_0 + h v'_0 + \frac{1}{2} h^2 v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)}}{g(t)!} \theta(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\ v'_1(t_0 + h) &= v'_0 + h v''_0 + \sum_{i=1}^l b'_i B(\vartheta_i''', v_0, v'_0) \\ &= v'_0 + h v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)-1}}{g(t)!} \theta'(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \\ v''_1(t_0 + h) &= v''_0 + \sum_{i=1}^l b''_i B(\vartheta_i''', v_0, v'_0) \\ &= v''_0 \\ &+ \sum_{t \in \mathbb{T}\mathbb{T}/\{Y_1, Y_2\}} \frac{h^{g(t)-2}}{g(t)!} \theta''(t) \alpha(t) \gamma(t) F(t)(v, v', v'') \end{aligned}$$

### 3.1 Algebraic order criteria(conditions)

We achieved the purpose of this research through two theorems (3.1) and (3.3). In (Table 2), the tri-colored trees of orders up to seven are listed together with the accompanying function values.

**Theorem 3.4:** The RKTG method (2) has order  $q$  if and only if these prerequisites are fulfilled

$$\theta(t) = \frac{1}{\gamma(t)}, \quad \text{for } g(t) \leq q$$

$$\theta'(t) = \frac{g(t)}{\gamma(t)}, \quad \text{for } g(t) \leq q + 1$$

$$\theta''(t) = \frac{g(t)(g(t) - 1)}{\gamma(t)}, \quad \text{for } g(t) \leq q + 2$$

**Corollary 3.1:** presuppose that

$$t^* = \{Y_1^{\theta_1+2\theta_2}, t_3^{\theta_3}, \dots, t_r^{\theta_r} < t_{r+1}^{\theta_{r+1}}, \dots, t_l^{\theta_l} > \}_3$$

$$\hat{t} = \{Y_1^{\theta_1}, Y_2^{\theta_2}, t_3^{\theta_3}, \dots, t_r^{\theta_r} < t_{r+1}^{\theta_{r+1}}, \dots, t_l^{\theta_l} > \}_3$$

where  $t_3, \dots, t_r$  unique and separate from  $Y_1$  and  $Y_2$ , and  $t_{r+1}, \dots, t_l$  unique then

$$\omega_i(\hat{t}) = \frac{1}{2\theta_2} \omega_i(t^*) \quad , \quad g(\hat{t}) = g_i(t^*) \quad , \quad \gamma(\hat{t}) = 2\theta_2 \gamma(t^*)$$

**Table 2. tri-colored trees with orders up to seven, elementary differentials, and their corresponding functions.**

| Order<br>$g(t)$ | $t$         | tree        | $\alpha(t)$ | Elementary weight | Density | $F(t)(v, v', v'')$   |
|-----------------|-------------|-------------|-------------|-------------------|---------|----------------------|
| 0               | $\emptyset$ | $\emptyset$ |             |                   |         |                      |
| 1               | $Y_1$       |             | 1           |                   | 1       | $v'$                 |
| 2               | $Y_2$       |             | 1           |                   | 2       | $v''$                |
| 3               | $Y_3$       |             | 1           | e                 | 6       | $f$                  |
| 4               | $t_{41}$    |             | 1           | c                 | 24      | $f_v v'$             |
| 4               | $t_{42}$    |             | 1           | c                 | 24      | $f_v v''$            |
| 5               | $t_{51}$    |             | 1           | $c^2$             | 60      | $f_{vv}(v')^2$       |
| 5               | $t_{52}$    |             | 2           | $c^2$             | 60      | $f_{vv} v' v''$      |
| 5               | $t_{53}$    |             | 1           | $c^2$             | 60      | $f_{v'v'}(v'')^2$    |
| 5               | $t_{54}$    |             | 1           | Ae                | 120     | $f_v f$              |
| 5               | $t_{55}$    |             | 1           | $\frac{1}{2} c^2$ | 120     | $f_v v''$            |
| 6               | $t_{61}$    |             | 1           | Ae                | 720     | $f_v f$              |
| 6               | $t_{62}$    |             | 1           | c                 | 720     | $f_v f_v v''$        |
| 6               | $t_{63}$    |             | 1           | c                 | 720     | $f_v f_v v'$         |
| 6               | $t_{64}$    |             | 3           | $\frac{1}{2} c^2$ | 240     | $f_{vv} v' v''$      |
| 6               | $t_{65}$    |             | 1           | $c^3$             | 120     | $f_{vvv}(v')^2$      |
| 6               | $t_{66}$    |             | 1           | $c^3$             | 120     | $f_{vvv} v''(v')^2$  |
| 6               | $t_{67}$    |             | 3           | $\frac{1}{2} c^2$ | 240     | $f_{vv}(v'')^2$      |
| 6               | $t_{68}$    |             | 3           | cAe               | 240     | $f_{vv} v' f$        |
| 6               | $t_{69}$    |             | 3           | $c^3$             | 120     | $f_{vvv}(v'')^2 v'$  |
| 6               | $t_{610}$   |             | 3           | cAe               | 240     | $f_{vv} v'' f$       |
| 6               | $t_{611}$   |             | 1           | $c^3$             | 120     | $f_{vvv}(v'')^3$     |
| 7               | $t_{71}$    |             | 4           | Ae                | 840     | $f_{vv} v' f$        |
| 7               | $t_{72}$    |             | 3           | $c^2$             | 840     | $f_{vv}(v'')^2$      |
| 7               | $t_{73}$    |             | 6           | $c^4$             | 420     | $f_{vvv}(v')^2 v''$  |
| 7               | $t_{74}$    |             | 12          | $c^4$             | 420     | $f_{vvv}(v'')^2 v'$  |
| 7               | $t_{75}$    |             | 1           | $c^4$             | 210     | $f_{vvvv}(v')^4$     |
| 7               | $t_{76}$    |             | 4           | $c^4$             | 210     | $f_{vvvv} v''(v')^3$ |
| 7               | $t_{77}$    |             | 10          | $Ae^2$            | 1260    | $f_{vv} v'' f$       |

|   |           |                                                                                   |    |                           |      |                       |
|---|-----------|-----------------------------------------------------------------------------------|----|---------------------------|------|-----------------------|
| 7 | $t_{78}$  |                                                                                   | 6  | $Ae^2$                    | 420  | $f_{vvv}(v')^2 f$     |
| 7 | $t_{79}$  |                                                                                   | 6  | $c^3$                     | 420  | $f_{vvv}(v'')^3$      |
| 7 | $t_{710}$ |                                                                                   | 4  | $c^2$                     | 2520 | $f_{vv} f_v(v')^2$    |
| 7 | $t_{711}$ |                                                                                   | 6  | $c^2$                     | 2520 | $f_{vv} f_v v' v''$   |
| 7 | $t_{712}$ |                                                                                   | 12 | Ac                        | 420  | $f_{vvv} v' v'' f$    |
| 7 | $t_{713}$ |                                                                                   | 6  | $c^4$                     | 210  | $f_{vvv}(v'')^2 (v')$ |
| 7 | $t_{714}$ |                                                                                   | 4  | $c^4$                     | 210  | $f_{vvv}(v'')^3 v'$   |
| 7 | $t_{715}$ |  | 3  | A                         | 5040 | $f_v v f^2$           |
| 7 | $t_{716}$ |                                                                                   | 4  | $Ae^2$                    | 1260 | $f_{vv} f_v v'' v'$   |
| 7 | $t_{717}$ |                                                                                   | 5  | $c^4$                     | 2520 | $f_{vv} f_v(v'')^2$   |
| 7 | $t_{718}$ |                                                                                   | 6  | $Ae^2$                    | 420  | $f_{vv} v'(v'')^2 f$  |
| 7 | $t_{719}$ |                                                                                   | 1  | $c^4$                     | 210  | $f_{vv} v'(v'')^4$    |
| 7 | $t_{720}$ |                                                                                   | 2  | $\frac{1}{2} \bar{A} c^2$ | 5040 | $f_v f_v v''$         |
| 7 | $t_{721}$ |                                                                                   | 1  | $c^2$                     | 2520 | $f_v f_{vv}(v')^2$    |
| 7 | $t_{722}$ |                                                                                   | 1  | AAe                       | 5040 | $f_v f_v f$           |
| 7 | $t_{723}$ |                                                                                   | 1  | c                         | 5040 | $f_v f_v v'$          |

In Corollary 3.1, it is stated that the two trees  $t^*$  and  $\hat{t}$  have the same order conditions, making them comparable. Next, in Table 2, we can reduce the number of trees because they are identical. For instance, in Table 2, trees  $t_{713}$  and  $t_{714}$  are identical. The order criteria for the RKTG formula to have orders ranging from one to seven can be expressed via the last Theorem and its Corollary.

**Order criteria for v**

$$\begin{aligned}
 3: \quad & \sum b_i = \frac{1}{6} \\
 4: \quad & \sum b_i c_i = \frac{1}{24} \\
 5: \quad & \sum b_i c_i^2 = \frac{1}{60} \quad , \quad \sum b_i a_{ij} = \frac{1}{120} \\
 6: \quad & \sum b_i \bar{a}_{ij} c_j = \frac{1}{720} \quad , \quad \sum b_i c_i^3 \\
 & = \frac{1}{120} \quad , \quad \sum b_i a_{ij} c_i \\
 & = \frac{1}{240} \quad , \quad \sum b_i a_{ij} = \frac{1}{720}
 \end{aligned}$$

$$\begin{aligned}
 7: \sum b_i c_i^3 a_{ij} &= \frac{1}{420}, \sum b_i c_i a_{ij} c_j \\
 &= \frac{1}{1260}, \sum b_i c_i^4 \\
 &= \frac{1}{210}, \sum b_i \bar{a}_{ij} c_j^2 = \frac{1}{2520} \\
 \sum b_i c_i a_{ij} &= \frac{1}{1260}, \sum b_i a_{ij} c_j \\
 &= \frac{1}{5040}, \sum b_i \bar{a}_{ij} a_{ik} \\
 &= \frac{1}{5040}, \sum b_i \bar{a}_{ij} \bar{a}_{ik} = \frac{1}{840}
 \end{aligned}$$

Order criteria for  $v'$ 

$$\begin{aligned}
 2: \sum \bar{b}_i &= \frac{1}{2} \\
 3: \sum \bar{b}_i c_i &= \frac{1}{6} \\
 4: \sum \bar{b}_i c_i^2 &= \frac{1}{12}, \sum \bar{b}_i \bar{a}_{ij} = \frac{1}{24} \\
 5: \sum \bar{b}_i c_i^3 &= \frac{1}{20}, \sum \bar{b}_i \bar{a}_{ij} c_j \\
 &= \frac{1}{120}, \sum \bar{b}_i c_i a_{ij} \\
 &= \frac{1}{40}, \sum \bar{b}_i a_{ij} = \frac{1}{120} \\
 6: \sum \bar{b}_i c_i^3 \bar{a}_{ij} &= \frac{1}{60}, \sum \bar{b}_i c_i a_{ij} c_j \\
 &= \frac{1}{180}, \sum \bar{b}_i c_i^4 \\
 &= \frac{1}{30}, \sum \bar{b}_i \bar{a}_{ij} c_j^2 = \frac{1}{360} \\
 \sum \bar{b}_i c_i a_{ij} &= \frac{1}{180}, \sum \bar{b}_i a_{ij} c_j \\
 &= \frac{1}{720}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} \\
 &= \frac{1}{720}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} = \frac{1}{120} \\
 7: \sum \bar{b}_i c_i^5 &= \frac{1}{42}, \sum \bar{b}_i c_i^2 \bar{a}_{ij} c_j \\
 &= \frac{1}{252}, \sum \bar{b}_i c_i^3 \bar{a}_{ij} \\
 &= \frac{1}{42}, \sum \bar{b}_i c_i a_{ij} c_j = \frac{1}{1008} \\
 \sum \bar{b}_i \bar{a}_{ij} c_j^3 &= \frac{1}{840}, \sum \bar{b}_i c_i \bar{a}_{ij} c_j^2 \\
 &= \frac{1}{504}, \sum \bar{b}_i c_i^2 a_{ij} \\
 &= \frac{1}{252}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} c_k = \frac{1}{5040}
 \end{aligned}$$

$$\begin{aligned}
 \sum \bar{b}_i a_{ij} \bar{a}_{ik} &= \frac{1}{504}, \sum \bar{b}_i a_{ij} c_j^2 \\
 &= \frac{1}{2520}, \sum \bar{b}_i c_i \bar{a}_{ij} \bar{a}_{ik} \\
 &= \frac{1}{168}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} c_j = \frac{1}{504} \\
 \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} c_k &= \frac{1}{504}
 \end{aligned}$$

Order criteria for  $v''$ 

$$\begin{aligned}
 1: \sum \bar{\bar{b}}_i &= 1 \\
 2: \sum \bar{\bar{b}}_i c_i &= \frac{1}{2} \\
 3: \sum \bar{\bar{b}}_i c_i^2 &= \frac{1}{3}, \sum \bar{\bar{b}}_i \bar{a}_{ij} = \frac{1}{6} \\
 4: \sum \bar{\bar{b}}_i c_i^3 &= \frac{1}{4}, \sum \bar{\bar{b}}_i \bar{a}_{ij} c_j = \frac{1}{24}, \sum \bar{\bar{b}}_i a_{ij} \\
 &= \frac{1}{24}, \sum \bar{\bar{b}}_i c_i a_{ij} = \frac{1}{8} \\
 5: \sum \bar{\bar{b}}_i c_i^4 &= \frac{1}{5}, \sum \bar{\bar{b}}_i \bar{a}_{ij} c_j^2 \\
 &= \frac{1}{60}, \sum \bar{\bar{b}}_i c_i a_{ij} \\
 &= \frac{1}{30}, \sum \bar{\bar{b}}_i \bar{a}_{ij} \bar{a}_{jk} = \frac{1}{120} \\
 \sum \bar{\bar{b}}_i \bar{a}_{ik} \bar{a}_{ij} &= \frac{1}{20}, \sum \bar{\bar{b}}_i a_{ij} c_j \\
 &= \frac{1}{120}, \sum \bar{\bar{b}}_i c_i \bar{a}_{ij} c_j \\
 &= \frac{1}{2}, \sum \bar{\bar{b}}_i c_i^2 \bar{a}_{ij} = \frac{1}{10} \\
 6: \sum \bar{\bar{b}}_i c_i^5 &= \frac{1}{6}, \sum \bar{\bar{b}}_i c_i^2 \bar{a}_{ij} c_j = \\
 &= \frac{1}{36}, \sum \bar{\bar{b}}_i c_i^3 \bar{a}_{ij} = \frac{1}{12}, \sum \bar{\bar{b}}_i c_i a_{ij} c_j = \\
 &= \frac{1}{144}, \sum \bar{\bar{b}}_i \bar{a}_{ij} c_j^3 = \frac{1}{120}, \sum \bar{\bar{b}}_i c_i \bar{a}_{ij} c_j^2 = \\
 &= \frac{1}{72}, \sum \bar{\bar{b}}_i \bar{a}_{ij} \bar{a}_{jk} c_k = \frac{1}{720}, \sum \bar{\bar{b}}_i \bar{a}_{ik} a_{ij} = \\
 &= \frac{1}{72}, \sum \bar{\bar{b}}_i a_{ij} c_j^2 = \frac{1}{360}, \sum \bar{\bar{b}}_i c_i \bar{a}_{ik} \bar{a}_{ij} = \\
 &= \frac{1}{24}, \sum \bar{\bar{b}}_i \bar{a}_{ik} \bar{a}_{ij} c_j = \frac{1}{72} \\
 7: \sum \bar{\bar{b}}_i c_i^6 &= \frac{1}{7}, \sum \bar{\bar{b}}_i a_{ij} c_i^3 = \\
 &= \frac{1}{14}, \sum \bar{\bar{b}}_i \bar{a}_{ik} \bar{a}_{ij} c_j = \frac{1}{84}, \sum \bar{\bar{b}}_i a_{ij} c_j = \\
 &= \frac{1}{168}, \sum \bar{\bar{b}}_i c_i^2 \bar{a}_{ik} \bar{a}_{ij} = \frac{1}{28}, \sum \bar{\bar{b}}_i c_i^3 \bar{a}_{ij} c_j = \\
 &= \frac{1}{42}, \sum \bar{\bar{b}}_i \bar{a}_{ik} a_{ij} c_j = \frac{1}{336}, \sum \bar{\bar{b}}_i a_{ij} \bar{a}_{jk} = \\
 &= \frac{1}{840}, \sum \bar{\bar{b}}_i c_i \bar{a}_{ij} c_j^3 = \frac{1}{140}, \sum \bar{\bar{b}}_i \bar{a}_{ij} c_j^3 = \\
 &= \frac{1}{120}, \sum \bar{\bar{b}}_i \bar{a}_{ij} \bar{a}_{jk} c_j^2 = \frac{1}{420}, \sum \bar{\bar{b}}_i \bar{a}_{ij} c_j^4 =
 \end{aligned}$$

$$\begin{aligned} \frac{1}{210}, \sum \bar{b}_i c_i \bar{a}_{ij} c_j^3 &= \frac{1}{140}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} c_j^2 = \\ \frac{1}{280}, \sum \bar{b}_i \bar{a}_{ik} \bar{a}_{ij} c_j^2 &= \frac{1}{168}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} \bar{a}_{ks} = \\ \frac{1}{5040}, \sum \bar{b}_i c_i \bar{a}_{ij} \bar{a}_{jk} c_k &= \frac{1}{840}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} = \\ \frac{1}{252}, \sum \bar{b}_i \bar{a}_{ik} \bar{a}_{ij} c_j &= \frac{1}{336}, \sum \bar{b}_i c_i \bar{a}_{ik} \bar{a}_{ij} = \\ \frac{1}{84}, \sum \bar{b}_i \bar{a}_{ij} c_j^3 &= \frac{1}{840}, \sum \bar{b}_i c_i \bar{a}_{ij} c_j^2 = \\ \frac{1}{360}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{jk} c_j &= \frac{1}{1680}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} \bar{a}_{is} = \\ \frac{1}{56}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} c_i &= \frac{1}{84}, \sum \bar{b}_i \bar{a}_{ik} \bar{a}_{ij} c_i = \\ \frac{1}{84}, \sum \bar{b}_i c_i^2 \bar{a}_{ik} \bar{a}_{ij} &= \frac{1}{28}, \sum \bar{b}_i \bar{a}_{ik} c_k \bar{a}_{ij} c_j = \\ \frac{1}{252}, \sum \bar{b}_i \bar{a}_{ij} \bar{a}_{ik} c_k &= \frac{1}{336} \end{aligned}$$

We can use the following simple steps to reduce the equations to be solved.  $\sum \bar{a}_{ij} = \frac{c_i^2}{2}$ .

#### 4. ZERO-STABILITY OF THE NEW TECHNIQUES

In this discussion, we examine the zero-stability of the newly developed approaches; to demonstrate the convergence and stability of multi-step approaches, we must establish their behavior at zero. We also examine zero-stability to establish the upper bound of the multi-step approaches. The initial characteristic polynomial for the RKTG method for Equations (2)-(6) is derived from the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n+1} \\ \bar{h} v'_{n+1} \\ \bar{h}^2 v''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_n \\ \bar{h} v'_n \\ \bar{h}^2 v''_n \end{bmatrix}$$

Where  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is the identity matrix

And the matrix  $A = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  represents the

coefficients of the vectors  $v_{n+1}, \bar{h} v'_{n+1}, \bar{h}^2 v''_{n+1}$

Then, the 1<sup>st</sup> characteristic factor of the new technique is

$$g(\xi) = \det[I\xi - A] = \begin{vmatrix} \xi - 1 & \xi - 1 & \xi - \frac{1}{2} \\ 0 & \xi - 1 & \xi - 1 \\ 0 & 0 & \xi - 1 \end{vmatrix}$$

Then,  $g(\xi) = (\xi - 1)^3$  By finding the roots of the characteristic polynomial  $(\xi - 1)^3 = 0$  then  $\xi =$

1,1,1 is the root of  $\xi$ . Therefore, the RKTG method is Zero-Stable since all the roots equal one.

#### 5. DEVELOPMENT OF THE RKTG TECHNIQUES

According to the order conditions stated above, we now present implicit RKTG Techniques. The local truncation error (LTE) for the qth-order RKTG technique is defined as

$$\|\mathcal{L}_s^{(q+1)}\| = \sqrt{\left(\sum_{i=1}^{n_{q+1}} (\mathcal{L}_i^{(q+1)})^2 + \sum_{i=1}^{n'_{q+1}} (\mathcal{L}'_i^{(q+1)})^2 + \sum_{i=1}^{n''_{q+1}} (\mathcal{L}''_i^{(q+1)})^2\right)}$$

Where

the

$\mathcal{L}_i^{(q+1)}$  is the LTE term for  $v$ ,  $\mathcal{L}'_i^{(q+1)}$  is the LTE for  $v'$  and  $\mathcal{L}''_i^{(q+1)}$

is the LTE for  $v''$   $\mathcal{L}_s^{(q+1)}$  is the global truncation error (GTE)

##### 5.1. A Four-Stage Three-Order RKTG Technique

This part explains the process of developing the four-stage RKTG method, which is third order. The approach is obtained by utilizing the algebraic order criteria up to the fourth order and making simpler hypotheses.  $\sum \bar{a}_{ij} = \frac{c_i^2}{2}$  Finally, we get:

$$\begin{aligned} S = \left\{ a_{21} = -\frac{3}{32}, a_{31} = -\frac{3}{32}, a_{32} = \frac{1}{4}, a_{33} \right. \\ = \frac{1}{8}, aa_{21} = aa_{21}, aa_{31} \\ = aa_{31}, aa_{32} = aa_{32}, \\ aa_{33} = -\frac{2}{3}aa_{21} - \frac{2}{3}aa_{31} - \frac{2}{3}aa_{32} + \frac{1}{24}, b_1 \\ = -\frac{1}{6}, b_2 = \frac{1}{2}, b_3 = \frac{1}{6}, c_1 \\ = \frac{1}{2}, c_2 = \frac{1}{4}, c_3 = \frac{3}{4}, \\ d_1 = -\frac{1}{24}, d_2 = \frac{3}{16}, d_3 = \frac{1}{48}, g_1 = -\frac{1}{3}, g_2 \\ = \frac{2}{3}, g_3 = \frac{2}{3} \left. \right\} \end{aligned}$$

By using the Butcher tableau (Table 1) and the elements for the four-stage, three-order RKTG method, it can be expressed as shown in (Table 3)

**Table 3. The DRKTG4 Method**

|               |                    |               |
|---------------|--------------------|---------------|
| $\frac{1}{2}$ | $-\frac{523}{600}$ | $\frac{1}{8}$ |
|---------------|--------------------|---------------|

|                 |                |                    |                    |                 |               |               |
|-----------------|----------------|--------------------|--------------------|-----------------|---------------|---------------|
| $\frac{1}{4}$   | $\frac{3}{5}$  | $-\frac{523}{600}$ | $-\frac{3}{32}$    | $\frac{1}{8}$   |               |               |
| $\frac{3}{4}$   | $\frac{7}{10}$ | $\frac{1}{7}$      | $-\frac{523}{600}$ | $-\frac{3}{32}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |
| $-\frac{1}{24}$ | $\frac{3}{16}$ | $\frac{1}{48}$     |                    |                 |               |               |
| $-\frac{1}{6}$  | $\frac{1}{2}$  | $\frac{1}{6}$      |                    |                 |               |               |
| $-\frac{1}{3}$  | $\frac{2}{3}$  | $\frac{2}{3}$      |                    |                 |               |               |

|                  |                     |                   |                   |                |                    |                  |                |               |
|------------------|---------------------|-------------------|-------------------|----------------|--------------------|------------------|----------------|---------------|
| $\frac{1}{4}$    | $-\frac{71}{400}$   | $\frac{3}{50}$    |                   |                | $-\frac{3}{32}$    | $\frac{1}{8}$    |                |               |
| $\frac{1}{5}$    | $\frac{1}{10}$      | $\frac{1}{500}$   | $\frac{3}{50}$    |                | $-\frac{39}{1000}$ | $-\frac{3}{500}$ | $\frac{1}{8}$  |               |
| $\frac{1}{1000}$ | $\frac{1293}{1000}$ | $\frac{17}{875}$  | $\frac{201}{200}$ | $\frac{3}{50}$ | $-\frac{117}{56}$  | $\frac{9}{28}$   | $\frac{15}{7}$ | $\frac{1}{8}$ |
| $\frac{1}{4}$    | $\frac{1}{3}$       | $-\frac{5}{12}$   | $0$               |                |                    |                  |                |               |
| $1$              | $\frac{8}{9}$       | $\frac{25}{18}$   | $0$               |                |                    |                  |                |               |
| $2$              | $\frac{32}{27}$     | $-\frac{125}{54}$ | $\frac{7}{54}$    |                |                    |                  |                |               |

## 5.2. A Five-Stage Three-Order RKTG Method

The five-stage, three-order RKTG Technique is obtained by using algebraic concepts for order criteria up to the fifth order.

$$\begin{aligned}
 S &= \left\{ a_{21} = -\frac{3}{32}, a_{31} = -\frac{39}{1000}, a_{32} \right. \\
 &= -\frac{3}{500}, a_{41} = -\frac{117}{56}, a_{42} \\
 &= \frac{9}{28}, a_{43} = \frac{15}{7}, a_{44} = \frac{1}{8}, \\
 aa_{21} &= -\frac{5}{4} aa_{31} - \frac{5}{4} aa_{32} \\
 &= -\frac{3}{10}, aa_{31} = aa_{31}, aa_{32} \\
 &= aa_{32}, \\
 aa_{41} &= -\frac{65}{7} aa_{31} - \frac{190}{7} aa_{32} - \frac{219}{140} \\
 &= -\frac{3}{5} aa_{43}, aa_{42} \\
 &= \frac{27}{70} + \frac{125}{7} aa_{32} - \frac{2}{5} aa_{43}, aa_{43} \\
 &= aa_{43}, \\
 aa_{44} &= 5aa_{31} + 5aa_{32} + \frac{11}{20}, b_1 = 1, b_2 \\
 &= \frac{8}{9}, b_3 = -\frac{25}{18}, b_4 = 0, c_1 \\
 &= \frac{1}{2}, c_2 = \frac{1}{4}, c_3 = \frac{2}{5}, \\
 c_4 &= 1, d_1 = \frac{1}{4}, d_2 = \frac{1}{3}, d_3 = -\frac{5}{12}, d_4 = 0, g_1 \\
 &= 2, g_2 = \frac{32}{27}, g_3 = -\frac{125}{54}, g_4 \\
 &= \frac{7}{54} \}
 \end{aligned}$$

By using the Butcher tableau (Table 1) and the components of the five-stage, three-order RKTG technique, it can be represented as demonstrated in Table 4)

**Table 4. The DRKTG5 Method**

|               |                |               |
|---------------|----------------|---------------|
| $\frac{1}{2}$ | $\frac{3}{50}$ | $\frac{1}{8}$ |
|---------------|----------------|---------------|

## 6. NUMERICAL EXPERIMENTS

This section explores several concerns about the formula.  $v''' = f(t, v, v')$ . The numerical findings are compared with outcomes obtained by converting the same set of cases into a first-order system and solving it using the existing Runge-Kutta technique of the equivalent order.

**DRKTG4:** The implicit DRKTG4 technique of order four with three stages derived in the present study

**DRKTG5:** The four-stage fifth-order implicit DRKTG5 approach determined in this work.

**RKLIIIB4:** Two-stage fourth-order implicit RK method as described in<sup>(9)</sup>.

**DIRKN4:** Three-stage fourth-order diagonally implicit method presented in<sup>(26)</sup>.

**RadauIA:** Three-stage fifth-order implicit RK method presented in<sup>(10)</sup>.

**RadauII:** Three-stage fifth-order implicit RK method presented in<sup>(27)</sup>.

Case 1 (Homogeneous Linear Problem)

$$\begin{aligned}
 v''' &= -25v' \quad v(0) = 0, \\
 v'(0) &= 0, \quad v''(0) = 1
 \end{aligned}$$

The exact solution of this case is given by

$$v_1(t) = \frac{1}{25} - \frac{1}{25} \cos(5t)$$

Case 2 (Nonhomogeneous Linear Problem)

$$\begin{aligned}
 v''' &= v' + \cos^2(t) - 1 \quad v(0) = 0, \\
 v'(0) &= 0, \quad v''(0) = 1
 \end{aligned}$$

The exact solution of this case is given by

$$v_1(t) = \frac{1}{5} e^{-t} + \frac{4}{5} e^t - \frac{1}{20} \sin(2t) + \frac{1}{2} t - 1$$

Case 3 (Homogeneous Nonlinear Problem)

$$\begin{aligned}
 v''' &= \frac{3v'}{4v^4} \quad v(0) = 1, \\
 v'(0) &= \frac{1}{2}, \quad v''(0) = \frac{-1}{4}
 \end{aligned}$$

The exact solution of this case is given by

$$v_1(t) = \sqrt{t+1}$$

Case 4 (Nonhomogeneous Nonlinear Problem)

$$v''' = v^2 + \cos^2(t) - v' - 1 \quad v(0) = 0, \\ v'(0) = 1, \quad v''(0) = 0$$

The exact solution of this case is given by

$$v_1(t) = \sin x$$

Case 5 (non-linear system)

$$v_1'''(t) = \frac{1}{2}e^{4t}v_3(t)v_2'(t)$$

$$v_2'''(t) = \frac{8}{3}e^{2t}v_1(t)v_3'(t)$$

$$v_3'''(t) = 27v_2(t)v_1'(t)$$

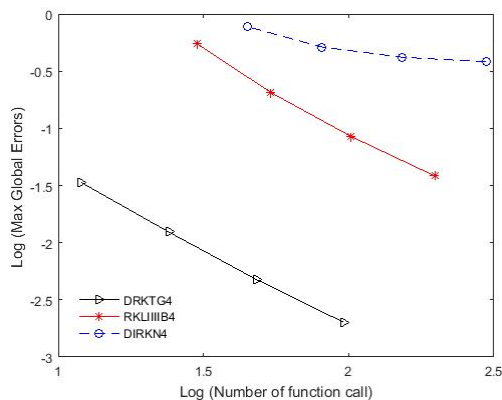
$$v_1(0) = 1, \quad v_1'(0) = -1, \quad v_1''(0) = 1$$

$$v_2(0) = 1, \quad v_2'(0) = -2, \quad v_2''(0) = 4$$

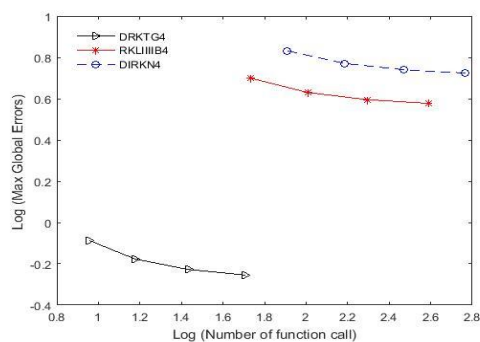
$$v_3(0) = 1, \quad v_3'(0) = -3, \quad v_3''(0) = 9$$

The exact solution of this case is given by

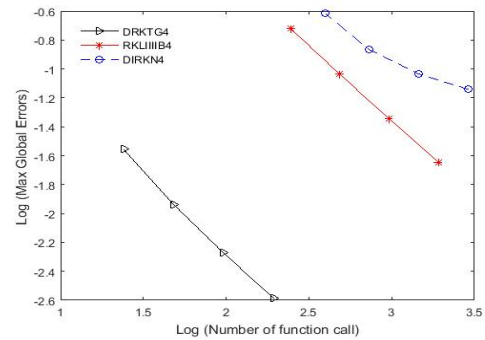
$$v_1(t) = e^{-t}, \quad v_2(t) = e^{-2t}, \quad v_3(t) = e^{-3t}$$



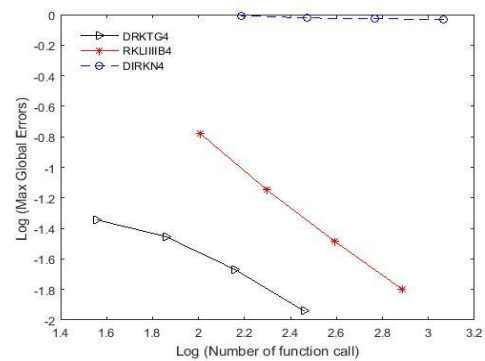
(a)



(b)



(c)



(d)

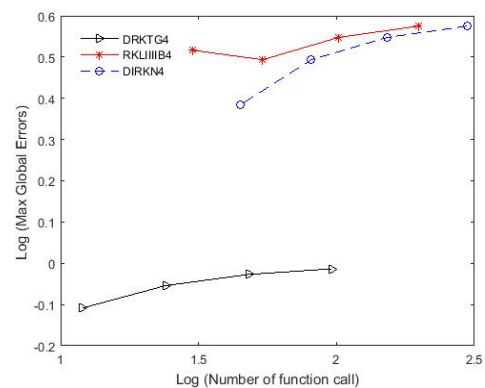
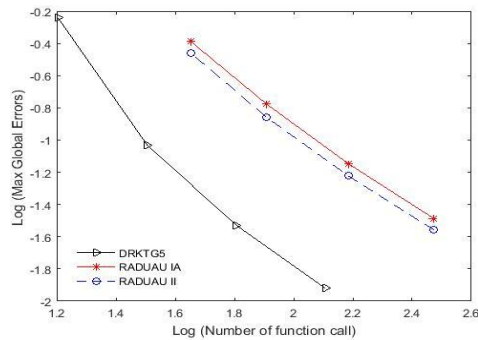


Fig. 1: Accuracy curve for DRK4G4, RK4IIB4,

DIRKN4

for Problems (1-5)



(a)

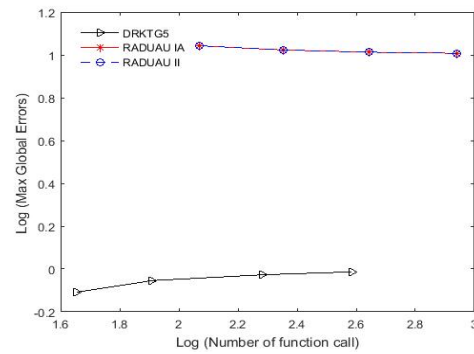
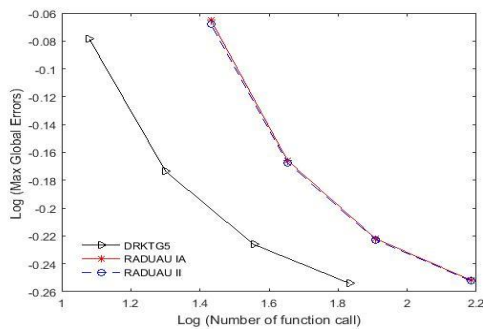
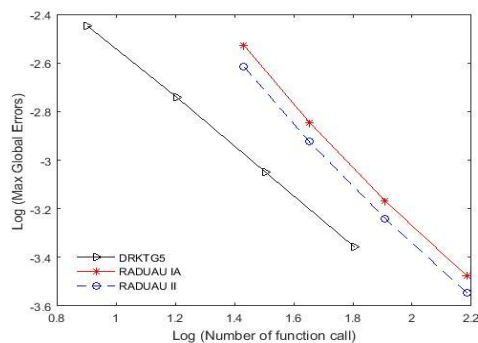


Fig. 2: Accuracy curve for DRKTG5, RadauIA,

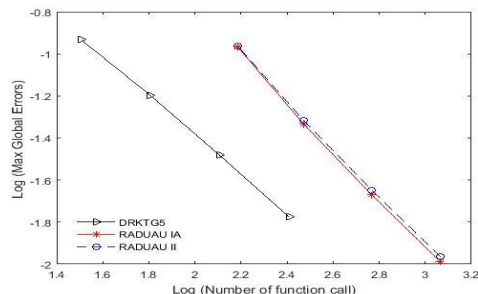
RadauII for Problems (1-5)



(b)



(c)



(d)

**Figures (1, 2):** The DRKTG technique presents the decimal logarithm of the maximum global error and the function evaluations. This provides a visual representation of the approach's effectiveness. The DRKTG5 technique requires fewer function evaluations compared to other implicit Runge-Kutta techniques of equivalent order. The number of equations increased threefold when the problems were transformed into first-order ODEs. Additionally, Figure (2) demonstrates that the DRKTG5 techniques exhibit the lowest maximum global error and require only a few function evaluations for all steps. DRKTG5 yields more precise results than any other methods documented in the scientific literature, as shown in Figures (1) and (2) (RadauIA and RadauII). In this study, we utilize steps of varying sizes ( $h = 0.25, 0.125, 0.0625$  and  $0.03125$ ), with the greatest global error across five test problems.

## 7. CONCLUSION

This study specifically examines the algebraic mathematical theory of order criteria for the RKTG technique. This technique is employed for the direct solution of third-order ordinary differential equations (ODEs) with special features. Numerous scholars have introduced the notion and concept of B-series theory, which relies on algebraic order conditions, as well as definitions and algebraic theories of order conditions.

In fact, the objective of the current study is to solve ODEs of order three directly by using the B-series and tri-colored tree theories to construct the RKTG

formula based on the algebraic order requirements given in the form of  $v''' = f(t, v, v')$ . Consequently, we devised a three-stage fourth-order technique, the DRKTG4 technique and a four-stage fifth-order technique, the DRKTG5 technique. Figures 1 and 2 graphically illustrate the numerical results of these techniques as proficiency curves. These charts show the correlation between the logarithm of the highest global error during integration and the associated computational cost, measured by the number of function evaluations.

The numerical findings show that the novel DRKTG methods are more efficient than current RK methods in the scientific literature in terms of the number of function evaluations required to solve the specific third-order ODEs.

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**Author contributions:** The authors contributed equally to the study.

## REFERENCES

- Panghal S, Kumar M. Optimization free neural network approach for solving ordinary and partial differential equations. *Engineering with Computers*. 2021;37(4):2989–3002.  
<https://link.springer.com/article/10.1007/s00366-020-00985-1>
- Danish GA, Imran M, Fetecau C, Vieru D. First exact solutions for mixed boundary value problems concerning the motions of fluids with exponential dependence of viscosity on pressure. *AIP Advances*. 2020;10(6).  
<https://doi.org/10.1063/1.5145369>
- Datsko B, Kutniv M, Kunynets A, Włoch A. New explicit high-order one-step methods for singular initial value problems. *Computational and Mathematical Methods*. 2021;3(3):e1099.  
<https://doi.org/10.1002/cmm4.1099>
- Ogunware BG, Abolarin OE, Kuboye JO, Adeyefa EO. New efficient numerical model for solving second, third and fourth order ordinary differential equations directly. *Gazi University Journal of Science*. 2020;33(4):821–33.  
<https://dergipark.org.tr/en/download/article-file/1184941>
- AL-Rabahi ZAA, Hasan YQ. Numerical Solutions of Fourth-order Singular Boundary Value Problems by New Modified Adomian Decomposition Method. 2020.
- Farooq U, Khan H, Tehier F, Hincal E, Baleanu D, Bin Jebreen H. New approximate analytical technique for the solution of time fractional fluid flow models. *Advances in Difference Equations*. 2021;2021:1–20.  
<https://link.springer.com/article/10.1186/s13662-021-03240-z>
- Ali RN, Gaftan AM. The Stability Of Crank–Nicholson and Explicit Methods for Numerical Solution for Sine–Gordon Equation. *Tikrit Journal of Pure Science*. 2023;28(4).  
<https://doi.org/10.25130/tjps.v28i4.1532>
- Khalaf KA, Khalaf BM. Techniques for Mixed Multiple Shooting for solve Stiff Initial Value Problems. *Tikrit Journal of Pure Science*. 2015;20(5):152–8.  
<https://doi.org/10.25130/tjps.v20i5.1251>
- Hairer E, Nørsett SP, Wanner G. Nonstiff problems: Springer; 1987.  
<https://library.wur.nl/WebQuery/titel/451015>
- Dormand JR. Numerical methods for differential equations: a computational approach: CRC press; 2018.  
<https://doi.org/10.1201/9781351075107>
- Lambert JD. Numerical methods for ordinary differential systems: the initial value problem: John Wiley & Sons, Inc.; 1991.  
<https://dl.acm.org/doi/abs/10.5555/129839>
- Kayode SJ. An Efficient Zero-Stable Numerical Method for Fourth-Order Differential Equations. *International Journal of Mathematics and Mathematical Sciences*. 2008;2008(1):IJMM364021.  
<https://doi.org/10.1155/2008/364021>
- Jimoh AK. Two-step hybrid block method for solving second order initial value problem of

- ordinary differential equations. *Earthline Journal of Mathematical Sciences*. 2024;14(5):1047–65. <https://doi.org/10.34198/ejms.14524.10471065>
14. Duromola M, Lawal R, Akinmoladun O. Numerical integration of linear hybrid multistep block method for third-order ordinary differential equations (IVPs). *Scientific African*. 2024:e02129. <https://doi.org/10.1016/j.sciaf.2024.e02129>
15. Waeleh N, Majid Z, Ismail F. A New Algorithm for Solving. *Applied Mathematical Sciences*. 2011;5(56):2795–805.
16. Hamza Y, Hamisu M, Buhari A. A new fixed coefficient diagonally implicit block backward differentiation formula for solving stiff initial value problems. *UMYU scientifica*. 2024;3(1):1–14. <https://doi.org/10.56919/usci.2431.001>
17. Fawzi FA, Globe HM, Ghawadri NG. Efficient Embedded Diagonal Implicit Runge-Kutta Method for Directly Solving Third Order ODEs. *Ibn AL-Haitham Journal For Pure and Applied Sciences*. 2024;37(2):404–26. <https://doi.org/10.30526/37.2.3409>
18. Lee KC, Senu N, Ahmadian A, Ibrahim SNI. On two-derivative Runge–Kutta type methods for solving  $u''' = f(x, u(x))$  with application to thin film flow problem. *Symmetry*. 2020;12(6):924. <https://doi.org/10.3390/sym12060924>
19. Alshareeda FAF. Runge-Kutta type methods for solving third-order ordinary differential equations and first-order oscillatory problems: Ph. D. thesis), Universiti Putra Malaysia; 2017. <http://psasir.upm.edu.my/id/eprint/70876>
20. Fawzi FA, Globe HM. Diagonally Implicit Type RK Method for Direct Solution of Third Order Ordinary Differential Equations Using Relevant-Colored Tree and the Associated B-series Theory. *Tikrit Journal for Computer Science and Mathematics*. 2023;1(1):1–20. <https://doi.org/10.25130/tjcsm.v1i1.3>
21. Saleh OM, Fawzi FA. Direct Solution of General Fourth-Order Ordinary Differential Equations Using the Diagonally Implicit Runge-Kutta Method. *Journal of Education for Pure Science*. 2023;13(1):11–21. <https://doi.org/10.32792/jeps.v13i1.249>
22. Ghawadri N, Senu N, Adel Fawzi F, Ismail F, Ibrahim ZB. Explicit Integrator of Runge-Kutta Type for Direct Solution of  $u(4) = f(x, u, u', u'')$ . *Symmetry*. 2019;11(2):246. <https://doi.org/10.3390/sym11020246>
23. Jumaa MH, Fawzi FA. The Implementations of the Embedded Diagonally Implicit Type Runge-Kutta Method (EDITRKM) For Special Third Order of the Ordinary Differential Equations. *Tikrit Journal of Pure Science*. 2022;27(3):85–91. <http://dx.doi.org/10.25130/tjps.27.2022.043>
24. Hairer E, Wanner G. A theory for Nyström methods. *Numerische Mathematik*. 1975;25:383–400. <https://link.springer.com/article/10.1007/BF01396335>
25. Chen Z, Qiu Z, Li J, You X. Two-derivative Runge-Kutta-Nyström methods for second-order ordinary differential equations. *Numerical Algorithms*. 2015;70:897–927. <https://link.springer.com/article/10.1007/s11075-015-9979-4>
26. Senu N, Suleiman M, Ismail F, Othman M. A new diagonally implicit Runge-Kutta-Nyström method for periodic IVPs. *WSEAS Trans Math*. 2010;9(9):679–88.
27. Butcher JC. *Numerical methods for ordinary differential equations*: John Wiley & Sons; 2016. <https://doi.org/10.1002/9781119121534>